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REMARKS ON THE NORMING SETS OF $\mathcal{L}(^m l_1^n)$ AND DESCRIPTION OF THE NORMING SETS OF $\mathcal{L}(^3 l_1^2)$

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Let $n\in\mathbb{N}, n\geq 2$. An element $x=(x_1,\dots,x_n)\in E^n$ is called a *norming point* of $T\in\mathcal{L}(^nE)$ if $\|x_1\|=\dots=\|x_n\|=1$ and $|T(x)|=\|T\|$, where $\mathcal{L}(^nE)$ denotes the space of all continuous n-linear forms on E. For $T\in\mathcal{L}(^nE)$ we define the *norming set* of T

$$\operatorname{Norm}(T)=\Big\{(x_1,\dots,x_n)\in E^n:(x_1,\dots,x_n)\text{ is a norming point of }T\Big\}.$$

By $i=(i_1,i_2,\dots,i_m)$ we denote the multi-index. In this paper we show the following:

(a) Let $n,m\geq 2$ and let $l_1^n=\mathbb{R}^n$ with the l_1-norm. Let $T=\big(a_i\big)_{1\leq i_k\leq n}\in\mathcal{L}(^m l_1^n)$ with $\|T\|=1$. Define $S=\big(b_i\big)_{1\leq i_k\leq n}\in\mathcal{L}(^n l_1^m)$ be such that $b_i=a_i$ if $|a_i|=1$ and $b_i=1$ if $|a_i|<1$.

Let $A=\{1,\dots,n\}\times\dots\times\{1,\dots,n\}$ and $M=\{i\in A:|a_i|<1\}$. Then,

$$\operatorname{Norm}(T)=\bigcup_{(i_1,\dots,i_m)\in M}\Big\{\Big(\big(t_1^{(1)},\dots,t_{i_1-1}^{(1)},0,t_{i_1+1}^{(1)},\dots,t_n^{(1)}\big),\big(t_1^{(2)},\dots,t_n^{(2)}\big),\dots,\big(t_1^{(m)},\dots,t_n^{(m)}\big)\Big),$$
$$\Big(\big(t_1^{(1)},\dots,t_n^{(1)}\big),\big(t_1^{(2)},\dots,t_{i_2-1}^{(2)},0,t_{i_2+1}^{(2)},\dots,t_n^{(2)}\big),\big(t_1^{(3)},\dots,t_n^{(3)}\big),\dots,\big(t_1^{(m)},\dots,t_n^{(m)}\big)\Big),\dots$$
$$\dots,\Big(\big(t_1^{(1)},\dots,t_n^{(1)}\big),\dots,\big(t_1^{(m-1)},\dots,t_n^{(m-1)}\big),\big(t_1^{(m)},\dots,t_{i_m-1}^{(m)},0,t_{i_m+1}^{(m)},\dots,t_n^{(m)}\big)\Big):$$
$$\Big(\big(t_1^{(1)},\dots,t_n^{(1)}\big),\dots,\big(t_1^{(m)},\dots,t_n^{(m)}\big)\Big)\in\operatorname{Norm}(S)\Big\}.$$

This statement extend the results of [9].

(b) Using the result (a), we describe the norming sets of every $T\in\mathcal{L}(^3 l_1^2)$.

1. Introduction. In 1961 Bishop and Phelps [2] showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, specially bounded linear operators between Banach spaces. The problem of denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [1], where they showed that the Radon-Nikodym property is sufficient for the denseness of norm attaining multilinear forms. Choi and Kim [3] showed that the Radon-Nikodym property is also sufficient for the denseness of norm attaining polynomials. Jiménez-Sevilla and Payá [5] studied the denseness of norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces.

Let $n\in\mathbb{N}, n\geq 2$. We write S_E for the unit sphere of a Banach space E. We denote by $\mathcal{L}(^nE)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\|=\sup_{x\in S_E\times\dots\times S_E}|T(x)|$. $\mathcal{L}_s(^nE)$ denote the closed subspace of all continuous symmetric n-linear forms on E. An element $(x_1,\dots,x_n)\in E^n$ is called a *norming point* of T if $\|x_1\|=\dots=\|x_n\|=1$ and $|T(x_1,\dots,x_n)|=\|T\|$.

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For $T \in \mathcal{L}(^nE)$, we define

$$\mathrm{Norm}(T) = \Big\{x = (x_1, \dots, x_n) \in E^n : x \text{ is a norming point of } T\Big\}.$$

$\mathrm{Norm}(T)$ is called the *norming set* of T.

Notice that $x = (x_1, \dots, x_n) \in \mathrm{Norm}(T)$ if and only if $(\epsilon_1 x_1, \dots, \epsilon_n x_n) \in \mathrm{Norm}(T)$ for some $\epsilon_k = \pm 1$ $(k = 1, \dots, n)$. Indeed, if $x \in \mathrm{Norm}(T)$, then

$$|T(\epsilon_1 x_1, \dots, \epsilon_n x_n)| = |\epsilon_1 \cdots \epsilon_n T(x_1, \dots, x_n)| = |T(x_1, \dots, x_n)| = \|T\|,$$

which shows that $(\epsilon_1 x_1, \dots, \epsilon_n x_n) \in \mathrm{Norm}(T)$. If $(\epsilon_1 x_1, \dots, \epsilon_n x_n) \in \mathrm{Norm}(T)$ for some $\epsilon_k = \pm 1$ $(k = 1, \dots, n)$, then $x = (x_1, \dots, x_n) = \Big(\epsilon_1(\epsilon_1 x_1), \dots, \epsilon_n(\epsilon_n x_n)\Big) \in \mathrm{Norm}(T)$.

The following examples show that $\mathrm{Norm}(T) = \varnothing$ or an infinite set.

Examples. (a) Let $T\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big) = \sum_{i=1}^{\infty} \frac{1}{2^i} x_i y_i \in \mathcal{L}_s(^2c_0)$. We claim that $\mathrm{Norm}(T) = \varnothing$. Obviously, $\|T\| = 1$. Assume that $\mathrm{Norm}(T) \neq \varnothing$. Let $\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big) \in \mathrm{Norm}(T)$. Then,

$$1 = \Big|T\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big)\Big| \le \sum_{i=1}^{\infty} \frac{1}{2^i}|x_i|\ |y_i| \le \sum_{i=1}^{\infty} \frac{1}{2^i} = 1,$$

which shows that $|x_i| = |y_i| = 1$ for all $i \in \mathbb{N}$. Hence, $(x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}} \notin c_0$. This is a contradiction. Therefore, $\mathrm{Norm}(T) = \varnothing$.

(b) Let $T\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big) = x_1 y_1 \in \mathcal{L}_s(^2c_0)$. Then,

$$\mathrm{Norm}(T) = \Big\{\Big((\pm 1, x_2, x_3, \dots),\ (\pm 1, y_2, y_3, \dots)\Big) \in c_0 \times c_0 : |x_j| \le 1,\ |y_j| \le 1 \text{ for } j \ge 2\Big\}.$$

A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a continuous n-linear form L on the product $E \times \cdots \times E$ such that $P(x) = L(x, \dots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$.

An element $x \in E$ is called a *norming point* of $P \in \mathcal{P}(^nE)$ if $\|x\| = 1$ and $|P(x)| = \|P\|$. For $P \in \mathcal{P}(^nE)$, we define $\mathrm{Norm}(P) = \Big\{x \in E : x \text{ is a norming point of } P\Big\}$. $\mathrm{Norm}(P)$ is called the *norming set* of P. Notice that $\mathrm{Norm}(P) = \varnothing$ or a finite set or an infinite set. Kim [7] classify $\mathrm{Norm}(P)$ for every $P \in \mathcal{P}(^2l_\infty^2)$, where $l_\infty^2 = \mathbb{R}^2$ with the supremum norm.

If $\mathrm{Norm}(T) \neq \varnothing$, $T \in \mathcal{L}(^nE)$ is called a *norm attaining* n-linear form and if $\mathrm{Norm}(P) \neq \varnothing$, $P \in \mathcal{P}(^nE)$ is called a *norm attaining* n-homogeneoue polynomial.(See [3])

For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [4].

It seems to be natural and interesting to study about $\mathrm{Norm}(T)$ for $T \in \mathcal{L}(^nE)$. For $m \in \mathbb{N}$, let $l_1^m := \mathbb{R}^m$ with the the l_1-norm and $l_\infty^2 = \mathbb{R}^2$ with the supremum norm. Notice that if $E = l_1^m$ or l_∞^2 and $T \in \mathcal{L}(^nE)$, $\mathrm{Norm}(T) \neq \varnothing$ since S_E is compact. Kim [6, 8, 10] classified $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}_s(^2l_\infty^2), \mathcal{L}(^2l_\infty^2), \mathcal{L}(^2l_1^2)$ or $\mathcal{L}_s(^2l_1^3)$. Kim [11] also classified $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}(^2\mathbb{R}^2_{h(w)})$, where $\mathbb{R}^2_{h(w)}$ denotes the plane with the hexagonal norm with weight $0 < w < 1$ $\|(x, y)\|_{h(w)} = \max\Big\{|y|, |x| + (1 - w)|y|\Big\}$. Recently, Kim [9] classified $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}_s(^3l_1^2)$.

In this paper we classify $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}(^3l_1^2)$. The main results of this article extend the results from [9].

2. The norming sets of $\mathcal{L}(^nl_1^m)$: auxiliary statements.

Lemma A ([10]). *Let $n, m \ge 2$. Let $T \in \mathcal{L}(^ml_1^n)$ with*

$$T\Big(\big(x_1^{(1)},\ldots,x_n^{(1)}\big),\ldots,\big(x_1^{(m)},\ldots,x_n^{(m)}\big)\Big)=\sum_{1\le i_k\le n,\ 1\le k\le m} a_{i_1\cdots i_m}\ x_{i_1}^{(1)}\cdots x_{i_m}^{(m)}$$

for some $a_{i_1\cdots i_m}\in\mathbb{R}$. *Then* $\|T\|=\max\{|a_{i_1\cdots i_m}|:1\le i_k\le n,\ 1\le k\le m\}$.

By simplicity we denote $T=\big(a_{i_1\cdots i_m}\big)_{1\le i_k\le n,1\le k\le m}$. We call $a_{i_1\cdots i_m}{}'$s the *coefficients* of T. Notice that if $\|T\|=1$, then $|a_{i_1\cdots i_m}|\le 1$ for all $1\le i_k\le n, 1\le k\le m$.

Lemma B [10]. *Let* $n,m\ge 2$ *and* $T\in\mathcal{L}({}^m l_1^n)$ *be the same as in Lemma A. If* $\Big(\big(t_1^{(1)},\ldots,t_n^{(1)}\big),\ldots,\big(t_1^{(m)},\ldots,t_n^{(m)}\big)\Big)\in\mathrm{Norm}(T)$ *and* $|a_{i_1'\cdots i_m'}|<\|T\|$ *for* $1\le i_k'\le n$, $1\le k\le m$, *then* $t_{i_1'}^{(1)}\cdot\ldots\cdot t_{i_m'}^{(m)}=0$.

Proposition C. *Let* $n,m\ge 2$ *and* $T=\big(a_{i_1\cdots i_m}\big)_{1\le i_k\le n,1\le k\le m}\in\mathcal{L}({}^m l_1^n)$ *be the same as in Lemma A with* $\|T\|=1$. *Let* $\delta_{i_1\cdots i_m}=1$ *if* $|a_{i_1\cdots i_m}|=1$ *and* $\delta_{i_1\cdots i_m}=0$ *if* $|a_{i_1\cdots i_m}|<1$. *Then,* $\mathrm{Norm}(T)=\mathrm{Norm}(T_\delta)$ *for* $T_\delta:=\big(a_{i_1\cdots i_m}\delta_{i_1\cdots i_m}\big)_{1\le i_k\le n,1\le k\le m}\in\mathcal{L}({}^m l_1^n)$.

Proof. ($\subseteq$). Let $(t^{(1)},t^{(2)},\ldots,t^{(m)})\in\mathrm{Norm}(T)$, where $t^{(k)}=(t_1^{(k)},t_2^{(k)},\ldots,t_n^{(k)})$ for $1\le k\le m$. Then

$$1=\Big|T\big(t^{(1)},\ldots,t^{(m)}\big)\Big|=\Big|\sum_{|a_{i_1'\cdots i_m'}|<1} a_{i_1'\cdots i_m'}\,t_{i_1'}^{(1)}\cdots t_{i_m'}^{(m)}+\sum_{|a_{i_1\cdots i_m}|=1} a_{i_1\cdots i_m}t_{i_1}^{(1)}\cdots t_{i_m}^{(m)}\Big|\overset{\text{by Lemma B}}{=}$$

$$=\Big|\sum_{|a_{i_1\cdots i_m}|=1} a_{i_1\cdots i_m}t_{i_1}^{(1)}\cdots t_{i_m}^{(m)}\Big|=\Big|\sum_{|a_{i_1'\cdots i_m'}|<1}\big(a_{i_1'\cdots i_m'}\delta_{i_1'\cdots i_m'}\big)t_{i_1'}^{(1)}\cdots t_{i_m'}^{(m)}+$$

$$+\sum_{|a_{i_1\cdots i_m}|=1}\big(a_{i_1\cdots i_m}\delta_{i_1\cdots i_m}\big)t_{i_1}^{(1)}\cdots t_{i_m}^{(m)}\Big|=\Big|T_\delta\big(t^{(1)},\ldots,t^{(m)}\big)\Big|.$$

($\supseteq$). Let $(t^{(1)},t^{(2)},\ldots,t^{(m)})\in\mathrm{Norm}(T_\delta)$. Write $x^{(k)}=\big(x_1^{(k)},\ldots,x_n^{(k)}\big)$ for $1\le k\le m$ and

$$T\big(x^{(1)},\ldots,x^{(m)}\big)=T_\delta\big(x^{(1)},\ldots,x^{(m)}\big)+\sum_{|a_{i_1\cdots i_m}|<1} a_{i_1\cdots i_m}\ x_{i_1}^{(1)}\cdots x_{i_m}^{(m)}.$$

Let $T_-\in\mathcal{L}({}^m l_1^n)$ be such that

$$T_-\big(x^{(1)},\ldots,x^{(m)}\big)=T_\delta\big(x^{(1)},\ldots,x^{(m)}\big)-\sum_{|a_{i_1\cdots i_m}|<1} a_{i_1\cdots i_m}\ x_{i_1}^{(1)}\cdots x_{i_m}^{(m)}.$$

By Lemma A, $\|T_-\|=1$. It follows that

$$1\ge\Big|T\big(t^{(1)},\ldots,t^{(m)}\big)\Big|=\Big|T_\delta\big(t^{(1)},\ldots,t^{(m)}\big)+\sum_{|a_{i_1\cdots i_m}|<1} a_{i_1\cdots i_m}\ t_{i_1}^{(1)}\cdots t_{i_m}^{(m)}\Big|,$$

$$1\ge\Big|T_-\big(t^{(1)},\ldots,t^{(m)}\big)\Big|=\Big|T_\delta\big(t^{(1)},\ldots,t^{(m)}\big)-\sum_{|a_{i_1\cdots i_m}|<1} a_{i_1\cdots i_m}\ t_{i_1}^{(1)}\cdots t_{i_m}^{(m)}\Big|,$$

which implies that

$$1\ge\Big|T_\delta\big(t^{(1)},\ldots,t^{(m)}\big)\Big|+\Big|\sum_{|a_{i_1\cdots i_m}|<1} a_{i_1\cdots i_m}\ t_{i_1}^{(1)}\cdots t_{i_m}^{(m)}\Big|=1+\Big|\sum_{|a_{i_1\cdots i_m}|<1} a_{i_1\cdots i_m}\ t_{i_1}^{(1)}\cdots t_{i_m}^{(m)}\Big|.$$

Thus, $\sum_{|a_{i_1\cdots i_m}|<1} a_{i_1\cdots i_m}\ t_{i_1}^{(1)}\cdots t_{i_m}^{(m)}=0$ and so

$$\Big|T\big(t^{(1)},\ldots,t^{(m)}\big)\Big|=\Big|T_\delta\Big(\big(t_1^{(1)},\ldots,t_n^{(1)}\big),\ldots,\big(t_1^{(m)},\ldots,t_n^{(m)}\big)\Big)\Big|=1.$$

□

The following shows that we can find $\mathrm{Norm}(T)$ for every $T\in\mathcal{L}(^n l_1^m)$ if we have known $\mathrm{Norm}(S)$ for every $S=\big(b_{i_1\cdots i_m}\big)_{1\le i_k\le n,1\le k\le m}\in\mathcal{L}(^n l_1^m)$ such that $\|S\|=1=|b_{i_1\cdots i_m}|$ for every $1\le i_k\le n, 1\le k\le m$.

Proposition D. *Let $n,m\geq 2$ and $T=\big(a_{i_1\cdots i_m}\big)_{1\le i_k\le n,1\le k\le m}\in\mathcal{L}(^m l_1^n)$ with $\|T\|=1$. Define $S=\big(b_{i_1\cdots i_m}\big)_{1\le i_k\le n,1\le k\le m}\in\mathcal{L}(^n l_1^m)$ be such that $b_{i_1\cdots i_m}=a_{i_1\cdots i_m}$ if $|a_{i_1\cdots i_m}|=1$ and $b_{i_1\cdots i_m}=1$ if $|a_{i_1\cdots i_m}|<1$.*

Let $A=\{1,\ldots,n\}\times\cdots\times\{1,\ldots,n\}$ and $M=\{(i_1,\ldots,i_m)\in A:|a_{i_1\cdots i_m}|<1\}$. Then,

$$\begin{aligned}&\mathrm{Norm}(T)=\bigcup_{(i_1,\ldots,i_m)\in M}\Big\{\Big(\big(t_1^{(1)},\ldots,t_{i_1-1}^{(1)},0,t_{i_1+1}^{(1)},\ldots,t_n^{(1)}\big),\big(t_1^{(2)},\ldots,t_n^{(2)}\big),\ldots,\big(t_1^{(m)},\ldots,t_n^{(m)}\big)\Big),\\&\Big(\big(t_1^{(1)},\ldots,t_n^{(1)}\big),\big(t_1^{(2)},\ldots,t_{i_2-1}^{(2)},0,t_{i_2+1}^{(2)},\ldots,t_n^{(2)}\big),\big(t_1^{(3)},\ldots,t_n^{(3)}\big),\ldots,\big(t_1^{(m)},\ldots,t_n^{(m)}\big)\Big),\\&\ldots,\Big(\big(t_1^{(1)},\ldots,t_n^{(1)}\big),\ldots,\big(t_1^{(m-1)},\ldots,t_n^{(m-1)}\big),\big(t_1^{(m)},\ldots,t_{i_m-1}^{(m)},0,t_{i_m+1}^{(m)},\ldots,t_n^{(m)}\big)\Big):\\&\Big(\big(t_1^{(1)},\ldots,t_n^{(1)}\big),\ldots,\big(t_1^{(m)},\ldots,t_n^{(m)}\big)\Big)\in\mathrm{Norm}(S)\Big\}.\end{aligned}$$

Proof. Let

$$\begin{aligned}\mathcal{F}\ =\ &\bigcup_{(i_1,\ldots,i_m)\in M}\Big\{\Big(\big(t_1^{(1)},\ldots,t_{i_1-1}^{(1)},0,t_{i_1+1}^{(1)},\ldots,t_n^{(1)}\big),\big(t_1^{(2)},\ldots,t_n^{(2)}\big),\ldots,\big(t_1^{(m)},\ldots,t_n^{(m)}\big)\Big),\\&\Big(\big(t_1^{(1)},\ldots,t_n^{(1)}\big),\big(t_1^{(2)},\ldots,t_{i_2-1}^{(2)},0,t_{i_2+1}^{(2)},\ldots,t_n^{(2)}\big),\big(t_1^{(3)},\ldots,t_n^{(3)}\big),\ldots,\big(t_1^{(m)},\ldots,t_n^{(m)}\big)\Big),\\&\ldots,\Big(\big(t_1^{(1)},\ldots,t_n^{(1)}\big),\ldots,\big(t_1^{(m-1)},\ldots,t_n^{(m-1)}\big),\big(t_1^{(m)},\ldots,t_{i_m-1}^{(m)},0,t_{i_m+1}^{(m)},\ldots,t_n^{(m)}\big)\Big):\\&\Big(\big(t_1^{(1)},\ldots,t_n^{(1)}\big),\ldots,\big(t_1^{(m)},\ldots,t_n^{(m)}\big)\Big)\in\mathrm{Norm}(S)\Big\}.\end{aligned}$$

We will show that $\mathrm{Norm}(T)=\mathcal{F}$. Note that $\|S\|=1$.

$(\subseteq)$. Let $\big(t^{(1)},\ldots,t^{(m)}\big):=\Big(\big(t_1^{(1)},\ldots,t_n^{(1)}\big),\ldots,\big(t_1^{(m)},\ldots,t_n^{(m)}\big)\Big)\in\mathrm{Norm}(T)$.

Since, by Lemma B, $t_{i_1'}^{(1)}\cdots t_{i_m'}^{(m)}=0$,

$$\begin{aligned}\Big|S\big(t^{(1)},\ldots,t^{(m)}\big)\Big|=\Big|&\sum_{|a_{i_1'\cdots i_m'}|<1}c_{i_1'\cdots i_m'}t_{i_1'}^{(1)}\cdots t_{i_m'}^{(m)}+\\+\sum_{|a_{i_1\cdots i_m}|=1}a_{i_1\cdots i_m}t_{i_1}^{(1)}\cdots t_{i_m}^{(m)}\Big|&=\Big|\sum_{|a_{i_1\cdots i_m}|=1}a_{i_1\cdots i_m}t_{i_1}^{(1)}\cdots t_{i_m}^{(m)}\Big|.\end{aligned}$$

Thus,

$$\begin{aligned}\Big|S\big(t^{(1)},\ldots,t^{(m)}\big)\Big|&=\Big|\sum_{|a_{i_1'\cdots i_m'}|<1}a_{i_1'\cdots i_m'}t_{i_1'}^{(1)}\cdots t_{i_m'}^{(m)}+\sum_{|a_{i_1\cdots i_m}|=1}a_{i_1\cdots i_m}t_{i_1}^{(1)}\cdots t_{i_m}^{(m)}\Big|=\\&=\Big|T\big(t^{(1)},\ldots,t^{(m)}\big)\Big|=1=\|S\|,\end{aligned}$$

thus $\big(t^{(1)},\ldots,t^{(m)}\big)\in\mathrm{Norm}(S)$ and $t_{i_1'}^{(1)}=0,\ldots,$ or $t_{i_m'}^{(m)}=0$ for $(i_1',\ldots,i_m')\in M$. Hence, $\big(t^{(1)},\ldots,t^{(m)}\big)\in\mathcal{F}$.

($\supseteq$). Let $\big(t^{(1)},\dots,t^{(m)}\big)\in\mathcal{F}$. It follows that

$$1=\Big|S\big(t^{(1)},\dots,t^{(m)}\big)\Big|=\Big|\sum_{|a_{i'_1\cdots i'_m}|<1}c_{i'_1\cdots i'_m}t^{(1)}_{i'_1}\cdots t^{(m)}_{i'_m}+\sum_{|a_{i_1\cdots i_m}|=1}a_{i_1\cdots i_m}t^{(1)}_{i_1}\cdots t^{(m)}_{i_m}\Big|=$$
$$=\Big[\text{since, } t^{(1)}_{i'_1}\cdots t^{(m)}_{i'_m}=0\Big]=\Big|\sum_{|a_{i_1\cdots i_m}|=1}a_{i_1\cdots i_m}t^{(1)}_{i_1}\cdots t^{(m)}_{i_m}\Big|=$$
$$=\Big|T_\delta\big(t^{(1)},\dots,t^{(m)}\big)\Big|=\Big[\text{by Proposition C}\Big]=\Big|T\big(t^{(1)},\dots,t^{(m)}\big)\Big|,$$

which implies that $\Big(\big(t^{(1)}_1,\dots,t^{(1)}_n\big),\dots,\big(t^{(m)}_1,\dots,t^{(m)}_n\big)\Big)\in\mathrm{Norm}(T)$. □

3. Classification of the norming sets of $\mathcal{L}(^3 l_1^2)$. In this section, we fully describe the norming sets of $\mathcal{L}(^3 l_1^2)$.

Lemma E. *Let $(x,y)\in S_{l_1^2}$. (1) If $|x\pm y|=1$, then $|x|=1$ or $|y|=1$. (2) Let $a,b\in\mathbb{R}$ be such that $0<|x|<1, |a|\le 1$ and $|b|\le 1$. If $1=|xa+yb|$, then $|a|=|b|=1$ and $sign(xy)=sign(ab)$.*

Proof. (1). Without loss of generality we may assume that $|x|\ge|y|$. Hence, $|x|-|y|=1=|x|+|y|$, $|x|=1, y=0$.

(2). *Claim 1.* $|a|=|b|=1$. Let's assume the contrary. Then $|a|<1$ or $|b|<1$. Notice that $|x|>0$ and $|y|>0$. It follows that $1=|xa+yb|\le|x||a|+|y||b|<|x|+|y|=1$, which is a contradiction. Thus the Claim 1 holds.

Claim 2. $\mathrm{sign}(xy)=\mathrm{sign}(ab)$. Again assume the contrary. Then $\mathrm{sign}(xy)=-\mathrm{sign}(ab)$. It follows that $1=|xa+yb|=\Big|x\,\mathrm{sign}(a)+y\,\mathrm{sign}(b)\Big|=\Big|\mathrm{sign}(y)\,\mathrm{sign}(a)(x\,\mathrm{sign}(a)+y\,\mathrm{sign}(b))\Big|=$ $=\Big||x|\,\mathrm{sign}(xy)+|y|\,\mathrm{sign}(ab)\Big|=\Big||x|-|y|\Big|\le\max\{|x|,|y|\}<1$, because $|x|<1,|y|<1$, which is a contradiction. Thus the Claim 2 holds. □

Let $T\in\mathcal{L}(^3 l_1^2)$ be of the form

$$T\big((x_1^{(1)},x_2^{(1)}),(x_1^{(2)},x_2^{(2)}),(x_1^{(3)},x_2^{(3)})\big)=a_{111}x_1^{(1)}x_1^{(2)}x_1^{(3)}+a_{222}x_2^{(1)}x_2^{(2)}x_2^{(3)}+a_{122}x_1^{(1)}x_2^{(2)}x_2^{(3)}+$$
$$+a_{212}x_2^{(1)}x_1^{(2)}x_2^{(3)}+a_{221}x_2^{(1)}x_2^{(2)}x_1^{(3)}+a_{211}x_2^{(1)}x_1^{(2)}x_1^{(3)}+a_{121}x_1^{(1)}x_2^{(2)}x_1^{(3)}+a_{112}x_1^{(1)}x_1^{(2)}x_2^{(3)}$$

for some $a_{ijk}\in\mathbb{R}$. For simplicity we denote $T=\big(a_{ijk}\big)_{i,j,k=1,2}$.

By Lemma A,

$$\|T\|=\max\big\{|a_{ijk}|:i,j,k=1,2\big\}.$$

Theorem F. *Let $T=\big(a_{ijk}\big)_{i,j,k=1,2}\in\mathcal{L}(^3 l_1^2)$ for some $a_{ijk}\in\mathbb{R}\in\mathbb{R}$. Then there exist $a^*_{ijk}\in\{\pm a_{ijk}:i,j,k=1,2\}$ such that $a^*_{111}\ge a^*_{222}\ge 0, a^*_{122}\ge 0, a^*_{212}\ge 0$ and*

$$\|T\|=\Big\|\big(a^*_{ijk}\big)_{i,j,k=1,2}\Big\|.$$

Proof. If $|a_{222}|>|a_{111}|$, we consider

$$T_1\big((x_1^{(1)},x_2^{(1)}),(x_1^{(2)},x_2^{(2)}),(x_1^{(3)},x_2^{(3)})\big)=T\big((x_2^{(1)},x_1^{(1)}),(x_2^{(2)},x_1^{(2)}),(x_2^{(3)},x_1^{(3)})\big).$$

Notice that $\|T_1\|=\|T\|$. Hence, we may assume that $|a_{111}|\ge|a_{222}|$. If $a_{111}<0$, we put

$$T_2\big((x_1^{(1)},x_2^{(1)}),(x_1^{(2)},x_2^{(2)}),(x_1^{(3)},x_2^{(3)})\big)=T\big((-x_1^{(1)},x_2^{(1)}),(x_1^{(2)},x_2^{(2)}),(x_1^{(3)},x_2^{(3)})\big).$$

Notice that $\|T_2\| = \|T\|$. Hence, we may assume that $a_{111} \geq 0$. If $a_{222} < 0$, we consider

$$T_3\big((x_1^{(1)}, x_2^{(1)}), (x_1^{(2)}, x_2^{(2)}), (x_1^{(3)}, x_2^{(3)})\big) = T\big((x_1^{(1)}, -x_2^{(1)}), (x_1^{(2)}, x_2^{(2)}), (x_1^{(3)}, x_2^{(3)})\big).$$

Notice that $\|T_3\| = \|T\|$. Hence, we may assume that $a_{222} \geq 0$. Thus we may assume that $a_{111} \geq a_{222} \geq 0$.

If $a_{122} < 0$, we put

$$T_5\big((x_1^{(1)}, x_2^{(1)}), (x_1^{(2)}, x_2^{(2)}), (x_1^{(3)}, x_2^{(3)})\big) = T\big((-x_1^{(1)}, x_2^{(1)}), (-x_1^{(2)}, x_2^{(2)}), (x_1^{(3)}, x_2^{(3)})\big).$$

Notice that $\|T_5\| = \|T\|$. Thus we may assume that $a_{122} \geq 0$. If $a_{212} < 0$, we put

$$T_6\big((x_1^{(1)}, x_2^{(1)}), (x_1^{(2)}, x_2^{(2)}), (x_1^{(3)}, x_2^{(3)})\big) = T\big((x_1^{(1)}, x_2^{(1)}), (x_1^{(2)}, -x_2^{(2)}), (x_1^{(3)}, -x_2^{(3)})\big).$$

Notice that $\|T_6\| = \|T\|$. Thus we may assume that $a_{212} \geq 0$. This completes the proof. □

Let $\mathcal{W} \subseteq S_{l_1^2} \times S_{l_1^2} \times S_{l_1^2}$. We denote

$$\mathrm{Sym}(\mathcal{W}) = \big\{(X,Y,Z), (X,Z,Y), (Y,X,Z), (Y,Z,X), (Z,X,Y), (Z,Y,X) : (X,Y,Z) \in \mathcal{W}\big\}.$$

By Theorem F, if $T = \big(a_{ijk}\big)_{i,j,k=1,2} \in \mathcal{L}(^3 l_1^2)$, then we may assume that $a_{111} \geq a_{222} \geq 0, a_{122} \geq 0, a_{212} \geq 0$.

We are in a position to classify the norming sets of $\mathcal{L}(^3 l_1^2)$.

Theorem G. *Let $T = \big(a_{ijk}\big)_{i,j,k=1,2} \in \mathcal{L}(^3 l_1^2)$ such that $\|T\| = 1$ with $a_{111} \geq a_{222} \geq 0, a_{122} \geq 0, a_{212} \geq 0$. The following assertions take place:*

Case 1. $|a_{ijk}| = 1$ *for all* $i, j, k = 1, 2$

1.1. $a_{ijk} = 1$ *for all* $i, j, k = 1, 2$

$$\mathrm{Norm}(T) = \Big\{\big(\pm(t, 1-t), \pm(s, 1-s), \pm(r, 1-r)\big) : 0 \leq t, s, r \leq 1\Big\}.$$

1.2. $1 = -a_{221} = a_{211} = a_{121} = a_{112}$

$$\begin{aligned}\mathrm{Norm}(T) = \Big\{&\big(\pm(1,0), \pm(t, 1-t), \pm(s, 1-s)\big), \big(\pm(t, 1-t), \pm(1,0), \pm(s, 1-s)\big),\\ &\big(\pm(t, 1-t), \pm(s, 1-s), \pm(0,1)\big), \big(\pm(0,1), \pm(t, -(1-t)), \pm(1,0)\big),\\ &\big(\pm(0,1), \pm(0,1), \pm(t, -(1-t))\big), \big(\pm(t, -(1-t)), \pm(0,1), \pm(1,0)\big) :\\ &0 \leq t, s \leq 1\Big\}.\end{aligned}$$

1.3. $1 = a_{221} = -a_{211} = a_{121} = a_{112}$

$$\begin{aligned}\mathrm{Norm}(T) = \Big\{&\big(\pm(1,0), \pm(t, 1-t), \pm(s, 1-s)\big), \big(\pm(t, 1-t), \pm(0,1), \pm(s, 1-s)\big),\\ &\big(\pm(t, 1-t), \pm(s, 1-s), \pm(0,1)\big), \big(\pm(0,1), \pm(0,1), \pm(t, -(1-t))\big),\\ &\big(\pm(t, -(1-t)), \pm(1,0), \pm(1,0)\big), \big(\pm(0,1), \pm(t, -(1-t)), \pm(1,0)\big) :\\ &0 \leq t, s \leq 1\Big\}.\end{aligned}$$

1.4. $1 = a_{221} = a_{211} = -a_{121} = a_{112}$

$$\begin{aligned}\mathrm{Norm}(T) &= \Big\{\big(\pm(0,1), \pm(t,1-t), \pm(s,1-s)\big), \big(\pm(t,1-t), \pm(1,0), \pm(s,1-s)\big),\\ &\quad \big(\pm(t,1-t), \pm(s,1-s), \pm(0,1)\big), \big(\pm(1,0), \pm(0,1), \pm(t,-(1-t))\big),\\ &\quad \big(\pm(t,-(1-t)), \pm(0,1), \pm(1,0)\big), \big(\pm(1,0), \pm(t,-(1-t)), \pm(1,0)\big) :\\ &\quad 0 \le t, s \le 1\Big\}.\end{aligned}$$

1.5. $1 = a_{221} = a_{211} = a_{121} = -a_{112}$

$$\begin{aligned}\mathrm{Norm}(T) &= \Big\{\big(\pm(0,1), \pm(t,1-t), \pm(s,1-s)\big), \big(\pm(t,1-t), \pm(0,1), \pm(s,1-s)\big),\\ &\quad \big(\pm(t,1-t), \pm(s,1-s), \pm(1,0)\big), \big(\pm(1,0), \pm(1,0), \pm(t,-(1-t))\big),\\ &\quad \big(\pm(t,-(1-t)), \pm(1,0), \pm(0,1)\big), \big(\pm(1,0), \pm(t,-(1-t)), \pm(0,1)\big) :\\ &\quad 0 \le t, s \le 1\Big\}.\end{aligned}$$

1.6. $1 = -a_{221} = -a_{211} = a_{121} = a_{112}$

$$\begin{aligned}\mathrm{Norm}(T) &= \Big\{\big(\pm(1,0), \pm(t,1-t), \pm(s,1-s)\big), \big(\pm(0,1), \pm(t,1-t), \pm(s,-(1-s))\big),\\ &\quad \big(\pm(t,1-t), \pm(s,1-s), \pm(0,1)\big), \big(\pm(t,-(1-t)), \pm(s,1-s), \pm(1,0)\big) :\\ &\quad 0 \le t, s \le 1\Big\}.\end{aligned}$$

1.7. $1 = -a_{221} = a_{211} = -a_{121} = a_{112}$

$$\begin{gathered}\mathrm{Norm}(T) = \Big\{\big(\pm(t,-(1-t)), \pm(1,0), \pm(s,-(1-s))\big),\\ \big(\pm(t,1-t), \pm(0,1), \pm(s,-(1-s))\big), \big(\pm(t,1-t), \pm(s,-(1-s)), \pm(1,0)\big),\\ \big(\pm(t,1-t), \pm(s,1-s), \pm(0,1)\big), \big(\pm(1,0), \pm(1,0), \pm(t,1-t)\big),\\ \big(\pm(0,1), \pm(0,1), \pm(t,1-t)\big) : 0 \le t, s \le 1\Big\}.\end{gathered}$$

1.8. $1 = -a_{221} = a_{211} = a_{121} = -a_{112}$

$$\begin{gathered}\mathrm{Norm}(T) = \Big\{\big(\pm(t,-(1-t)), \pm(1,0), \pm(0,1)\big), \big(\pm(t,-(1-t)), \pm(0,1), \pm(1,0)\big),\\ \big(\pm(t,1-t), \pm(1,0), \pm(1,0)\big), \big(\pm(t,1-t), \pm(0,1), \pm(0,1)\big),\\ \big(\pm(1,0), \pm(1,0), \pm(t,-(1-t))\big), \big(\pm(0,1), \pm(0,1), \pm(t,-(1-t))\big),\\ \big(\pm(1,0), \pm(0,1), \pm(t,1-t)\big), \big(\pm(0,1), \pm(1,0), \pm(t,1-t)\big),\\ \big(\pm(1,0), \pm(t,1-t), \pm(1,0)\big), \big(\pm(1,0), \pm(t,-(1-t)), \pm(0,1)\big),\\ \big(\pm(0,1), \pm(t,1-t), \pm(0,1)\big), \big(\pm(0,1), \pm(t,-(1-t)), \pm(1,0)\big) : 0 \le t, s \le 1\Big\}.\end{gathered}$$

1.9. $1 = a_{221} = -a_{211} = -a_{121} = a_{112}$

$$\begin{gathered}\mathrm{Norm}(T) = \Big\{\big(\pm(t,-(1-t)), \pm(s,-(1-s)), \pm(1,0)\big),\\ \big(\pm(t,1-t), \pm(s,1-s), \pm(0,1)\big), \big(\pm(1,0), \pm(1,0), \pm(t,1-t)\big),\\ \big(\pm(1,0), \pm(0,1), \pm(t,-(1-t))\big), \big(\pm(0,1), \pm(1,0), \pm(t,-(1-t))\big),\\ \big(\pm(0,1), \pm(0,1), \pm(t,1-t)\big) : {}''0 \le t, s \le 1\Big\}''.\end{gathered}$$

1.10. $1 = a_{221} = -a_{211} = a_{121} = -a_{112}$

$$\mathrm{Norm}(T) = \Big\{\big(\pm (t, -(1-t)), \pm(1,0), \pm(s, -(1-s))\big),$$
$$\big(\pm (t, 1-t), \pm(0,1), \pm(s, 1-s)\big), \big(\pm (1,0), \pm(t, 1-t), \pm(1,0)\big),$$
$$\big(\pm (1,0), \pm(t, -(1-t)), \pm(0,1)\big), \big(\pm (0,1), \pm(t, -(1-t)), \pm(1,0)\big),$$
$$\big(\pm (0,1), \pm(t, 1-t), \pm(0,1)\big) : 0 \le t, s \le 1\Big\}.$$

1.11. $1 = a_{221} = a_{211} = -a_{121} = -a_{112}$

$$\mathrm{Norm}(T) = \Big\{\big(\pm (1,0), \pm(t, -(1-t)), \pm(s, -(1-s))\big),$$
$$\big(\pm (0,1), \pm(t, 1-t), \pm(s, 1-s)\big), \big(\pm (t, 1-t), \pm(1,0), \pm(1,0)\big),$$
$$\big(\pm (t, -(1-t)), \pm(1,0), \pm(0,1)\big), \big(\pm (t, -(1-t)), \pm(0,1), \pm(1,0)\big),$$
$$\big(\pm (t, 1-t), \pm(0,1), \pm(0,1)\big) : 0 \le t, s \le 1\Big\}.$$

1.12. $1 = -a_{221} = -a_{211} = -a_{121} = a_{112}$

$$\mathrm{Norm}(T) = \Big\{\big(\pm (0,1), \pm(t, 1-t), \pm(s, -(1-s))\big),$$
$$\big(\pm (t, 1-t), \pm(0,1), \pm(s, -(1-s))\big), \big(\pm (t, 1-t), \pm(s, 1-s), \pm(0,1)\big),$$
$$\big(\pm (1,0), \pm(1,0), \pm(t, 1-t)\big), \big(\pm (1,0), \pm(t, -(1-t)), \pm(1,0)\big),$$
$$\big(\pm (t, -(1-t)), \pm(1,0), \pm(1,0)\big) \colon 0 \le t, s \le 1\Big\}.$$

1.13. $1 = -a_{221} = -a_{211} = a_{121} = -a_{112}$

$$\mathrm{Norm}(T) = \Big\{\big(\pm (0,1), \pm(t, 1-t), \pm(s, -(1-s))\big),$$
$$\big(\pm (t, -(1-t)), \pm(1,0), \pm(s, -(1-s))\big), \big(\pm (t, -(1-t)), \pm(s, 1-s), \pm(1,0)\big),$$
$$\big(\pm (1,0), \pm(0,1), \pm(t, 1-t)\big), \big(\pm (1,0), \pm(t, -(1-t)), \pm(0,1)\big),$$
$$\big(\pm (t, 1-t), \pm(0,1), \pm(0,1)\big) \colon 0 \le t, s \le 1\Big\}.$$

1.14. $1 = -a_{221} = a_{211} = -a_{121} = -a_{112}$

$$\mathrm{Norm}(T) = \Big\{\big(\pm (1,0), \pm(t, -(1-t)), \pm(s, -(1-s))\big),$$
$$\big(\pm (0,1), \pm(t, 1-t), \pm(0,1)\big), \big(\pm (t, 1-t), \pm(s, -(1-s)), \pm(1,0)\big),$$
$$\big(\pm (t, -(1-t)), \pm(1,0), \pm(0,1)\big), \big(\pm (t, 1-t), \pm(0,1), \pm(s, -(1-s)\big),$$
$$\big(\pm (1,0), \pm(1,0), \pm(t, -(1-t))\big) \colon 0 \le t, s \le 1\Big\}\text{''}.$$

1.15. $1 = a_{221} = -a_{211} = -a_{121} = -a_{112}$

$$\mathrm{Norm}(T) = \mathrm{Sym}\,\Big(\Big\{\big(\pm (1,0), \pm(t, -(1-t)), \pm(s, -(1-s))\big),$$
$$\big(\pm (0,1), \pm(0,1), \pm(t, 1-t)\big) \colon 0 \le t, s \le 1\Big\}\Big).$$

1.16. $1=-a_{221}=-a_{211}=-a_{121}=-a_{112}$

$$\text{Norm}(T)=\Big\{\big(\pm(1,0),\pm(t,-(1-t)),\pm(s,-(1-s))\big),$$
$$\big(\pm(0,1),\pm(t,1-t),\pm(s,-(1-s))\big),\big(\pm(t,1-t),\pm(0,1),\pm(s,-(1-s))\big),$$
$$\big(\pm(t,-(1-t)),\pm(1,0),\pm(s,-(1-s))\big):0\le t,s\le 1\Big\}\text{”}.$$

Case 2. $|a_{ijk}|<1$ *for some* $i,j,k=1,2$

Let $M=\{(i,j,k):|a_{ijk}|<1\}$ *and define* $S=\big(b_{ijk}\big)_{i,j,k=1,2}\in\mathcal{L}(^3l_1^2)$ *be such that* $b_{ijk}=a_{ijk}$ *if* $(i,j,k)\notin M$ *and* $b_{ijk}=1$ *if* $(i,j,k)\in M$. *(Notice that S is included in Case 1.)*

$$\textit{Then, }\text{Norm}(T)=\bigcup_{(i,j,k)\in M}\Big\{\big(\pm e_i,(t_1^{(2)},t_2^{(2)}),(t_1^{(3)},t_2^{(3)})\big),\big((t_1^{(1)},t_2^{(1)}),\pm e_j,(t_1^{(3)},t_2^{(3)})\big),$$
$$\big((t_1^{(1)},t_2^{(1)}),(t_1^{(2)},t_2^{(2)}),\pm e_k,\big):\big((t_1^{(1)},t_2^{(1)}),(t_1^{(2)},t_2^{(2)}),(t_1^{(3)},t_2^{(3)})\big)\in\text{Norm}(S)\Big\},$$

where $e_1=(1,0),e_2=(0,1)$.

Proof. Case 1.

Note that $1=a_{111}=a_{222}=a_{122}=a_{212}$ and

$$T\big((x_1^{(1)},x_2^{(1)}),(x_1^{(2)},x_2^{(2)}),(x_1^{(3)},x_2^{(3)})\big)=x_1^{(1)}x_1^{(2)}x_1^{(3)}+x_2^{(1)}x_2^{(2)}x_2^{(3)}+x_1^{(1)}x_2^{(2)}x_2^{(3)}+$$
$$+x_2^{(1)}x_1^{(2)}x_2^{(3)}+a_{221}x_2^{(1)}x_2^{(2)}x_1^{(3)}+a_{211}x_2^{(1)}x_1^{(2)}x_1^{(3)}+a_{121}x_1^{(1)}x_2^{(2)}x_1^{(3)}+a_{112}x_1^{(1)}x_1^{(2)}x_2^{(3)}.$$

We should consider sixteen subcases such that $a_{221}=\pm1,a_{211}=\pm1,a_{121}=\pm1,a_{112}=\pm1$.

Let $\big((t_1^{(1)},t_2^{(1)}),(t_1^{(2)},t_2^{(2)}),(t_1^{(3)},t_2^{(3)})\big)\in\text{Norm}(T)$. Without loss of generality we may assume that $t_1^{(k)}\ge0$ for every $k=1,2,3$. It follows that

$$1=\big|T\big((t_1^{(1)},t_2^{(1)}),(t_1^{(2)},t_2^{(2)}),(t_1^{(3)},t_2^{(3)})\big)\big|=\big|t_1^{(1)}\big|\,\big|t_1^{(3)}(t_1^{(2)}+a_{121}t_2^{(2)})+t_2^{(3)}(a_{112}t_1^{(2)}+t_2^{(2)})\big|+$$
$$+\big|t_2^{(1)}\big|\,\big|t_1^{(3)}(a_{211}t_1^{(2)}+a_{221}t_2^{(2)})+t_2^{(3)}(t_1^{(2)}+_2^{(2)})\big|.$$

Suppose that $t_1^{(1)}=1$.

Then,

$$1=\big|t_1^{(3)}\big(t_1^{(2)}+a_{121}t_2^{(2)})+t_2^{(3)}(a_{112}t_1^{(2)}+t_2^{(2)}\big)\big|\big|t_1^{(3)}\big|\,\big|t_1^{(2)}+a_{121}t_2^{(2)}\big|+\big|t_2^{(3)}\big|\,\big|a_{112}t_1^{(2)}+t_2^{(2)}\big|.$$

If $t_1^{(3)}=1$, then $\big|t_1^{(2)}+a_{121}t_2^{(2)}\big|=1$. Thus

$$\big((t_1^{(1)},t_2^{(1)}),(t_1^{(2)},t_2^{(2)}),(t_1^{(3)},t_2^{(3)})\big)=\big((1,0),(t,a_{121}(1-t)),(1,0)\big)\text{ for }\ 0\le t\le1.$$

If $t_1^{(3)}=0$, then $t_2^{(3)}=\big|a_{112}t_1^{(2)}+t_2^{(2)}\big|=1$. Thus

$$\big((t_1^{(1)},t_2^{(1)}),(t_1^{(2)},t_2^{(2)}),(t_1^{(3)},t_2^{(3)})\big)=\big((1,0),(t,a_{112}(1-t)),(0,1)\big)\text{ for }\ 0\le t\le1.$$

If $0<t_1^{(3)}<1$, then $0<t_2^{(3)}<1$ and by Lemma E(2),

$$1=\big|t_1^{(2)}+a_{121}t_2^{(2)}\big|=\big|a_{112}t_1^{(2)}+t_2^{(2)}\big|. \tag{1}$$

Suppose that $t_1^{(1)} = 0$.

Then, $t_2^{(1)} = 1$ and

$$1 = \left|t_1^{(3)}(a_{211}t_1^{(2)} + a_{221}t_2^{(2)}) + t_2^{(3)}(t_1^{(2)} + t_2^{(2)})\right| = \left|t_1^{(3)}\right| \left|a_{211}t_1^{(2)} + a_{221}t_2^{(2)}\right| + \left|t_2^{(3)}\right| \left|t_1^{(2)} + t_2^{(2)}\right|.$$

If $t_1^{(3)} = 1$, then $\left|a_{211}t_1^{(2)} + a_{221}t_2^{(2)}\right| = 1$. Thus

$$\big((t_1^{(1)}, t_2^{(1)}), (t_1^{(2)}, t_2^{(2)}), (t_1^{(3)}, t_2^{(3)})\big) = \big((0, 1), (a_{211}t, a_{221}(1 - t)), (1, 0)\big) \text{ for } 0 \le t \le 1.$$

If $t_1^{(3)} = 0$, then $t_2^{(3)} = \left|t_1^{(2)} + t_2^{(2)}\right| = 1$. Thus

$$\big((t_1^{(1)}, t_2^{(1)}), (t_1^{(2)}, t_2^{(2)}), (t_1^{(3)}, t_2^{(3)})\big) = \big((1, 0), (t, 1 - t), (0, 1)\big) \text{ for } 0 \le t \le 1.$$

If $0 < t_1^{(3)} < 1$, then $0 < t_2^{(3)} < 1$ and by Lemma E(2),

$$1 = \left|a_{211}t_1^{(2)} + a_{221}t_2^{(2)}\right| = \left|t_1^{(2)} + t_2^{(2)}\right|. \tag{2}$$

Suppose that $0 < t_1^{(1)} < 1$.

Then $0 < t_2^{(1)} < 1$ and by Lemma E(2),

$$1 = \left|t_1^{(3)}(t_1^{(2)} + a_{121}t_2^{(2)}) + t_2^{(3)}(a_{112}t_1^{(2)} + t_2^{(2)})\right| = \left|t_1^{(3)}(a_{211}t_1^{(2)} + a_{221}t_2^{(2)}) + t_2^{(3)}(t_1^{(2)} + t_2^{(2)})\right|. \tag{3}$$

We will only give the proof of 1.8 because those of the other subcases are similar.

Let $1 = -a_{221} = a_{211} = a_{121} = -a_{112}$.

Suppose that $t_1^{(1)} = 1$ and $0 < t_1^{(3)} < 1$.

By (1), $1 = \left|t_1^{(2)} + t_2^{(2)}\right| = \left|-t_1^{(2)} + t_2^{(2)}\right|$. By Lemma E(1), $t_1^{(2)} = 1$ or $\left|t_2^{(2)}\right| = 1$. If $t_1^{(2)} = 1$, then $1 = |t_1^{(3)} - t_2^{(3)}|$ and

$$\big((t_1^{(1)}, t_2^{(1)}), (t_1^{(2)}, t_2^{(2)}), (t_1^{(3)}, t_2^{(3)})\big) = \big((1, 0), (1, 0), (t, -(1 - t))\big) \text{ for } 0 \le t \le 1.$$

If $\left|t_2^{(2)}\right| = 1$, then $1 = |t_1^{(3)} + t_2^{(3)}|$ and

$$\big((t_1^{(1)}, t_2^{(1)}), (t_1^{(2)}, t_2^{(2)}), (t_1^{(3)}, t_2^{(3)})\big) = \big((1, 0), (0, 1), (t, 1 - t)\big) \text{ for } 0 \le t \le 1.$$

Suppose that $t_1^{(1)} = 0$ and $0 < t_1^{(3)} < 1$.

By (2), $1 = \left|t_1^{(2)} - t_2^{(2)}\right| = \left|t_1^{(2)} + t_2^{(2)}\right|$. If $t_1^{(2)} = 1$, then $1 = |t_1^{(3)} + t_2^{(3)}|$ and

$$\big((t_1^{(1)}, t_2^{(1)}), (t_1^{(2)}, t_2^{(2)}), (t_1^{(3)}, t_2^{(3)})\big) = \big((0, 1), (1, 0), (t, 1 - t)\big) \text{ for } 0 \le t \le 1.$$

If $\left|t_2^{(2)}\right| = 1$, then $1 = |t_1^{(3)} - t_2^{(3)}|$ and

$$\big((t_1^{(1)}, t_2^{(1)}), (t_1^{(2)}, t_2^{(2)}), (t_1^{(3)}, t_2^{(3)})\big) = \big((0, 1), (0, 1), (t, -(1 - t))\big) \text{ for } 0 \le t \le 1.$$

Suppose that $0 < t_1^{(1)} < 1$.

By (3), $1 = \left|t_1^{(3)}(t_1^{(2)} + t_2^{(2)}) + t_2^{(3)}(-t_1^{(2)} + t_2^{(2)})\right| = \left|t_1^{(3)}(t_1^{(2)} - t_2^{(2)}) + t_2^{(3)}(t_1^{(2)} + t_2^{(2)})\right|$. By Lemma E (1), $\left|t_1^{(2)}t_1^{(3)} + t_2^{(2)}t_2^{(3)}\right| = 1$ or $\left|t_1^{(3)}t_2^{(2)} - t_1^{(2)}t_2^{(3)}\right| = 1$.

Let $\left|t_1^{(2)}t_1^{(3)} + t_2^{(2)}t_2^{(3)}\right| = 1$. Then $t_1^{(2)}t_1^{(3)} = 1$ or $\left|t_2^{(2)}t_2^{(3)}\right| = 1$. If $t_1^{(2)}t_1^{(3)} = 1$ we get $\left|t_1^{(1)} - t_2^{(1)}\right| = 1$. Thus $\big((t_1^{(1)}, t_2^{(1)}), (t_1^{(2)}, t_2^{(2)}), (t_1^{(3)}, t_2^{(3)})\big) = \big((t, -(1 - t)), (1, 0), (1, 0)\big)$ for $0 \le t \le 1$.

If $\left|t_2^{(2)}t_2^{(3)}\right| = 1$, then $1 = \left|t_1^{(1)} + t_2^{(1)}\right| = 1$. Therefore,

$$\big((t_1^{(1)}, t_2^{(1)}), (t_1^{(2)}, t_2^{(2)}), (t_1^{(3)}, t_2^{(3)})\big) = \big((t, 1 - t), (0, 1), (0, 1)\big) \text{ for } 0 \le t \le 1.$$

Let $\left|t_1^{(3)}t_2^{(2)} - t_1^{(2)}t_2^{(3)}\right| = 1$. Then $\left|t_1^{(3)}t_2^{(2)}\right| = 1$ or $\left|t_1^{(2)}t_2^{(3)}\right| = 1$. If $\left|t_1^{(3)}t_2^{(2)}\right| = 1$ we have $1 = \left|t_1^{(1)} - t_2^{(1)}\right| = 1$. Thus

$$\big((t_1^{(1)}, t_2^{(1)}), (t_1^{(2)}, t_2^{(2)}), (t_1^{(3)}, t_2^{(3)})\big) = \big((t, -(1-t)), (0,1), (1,0)\big) \text{ for } 0 \le t \le 1.$$

If $\left|t_1^{(2)}t_2^{(3)}\right| = 1$ one has $\left|t_1^{(1)} - t_2^{(1)}\right| = 1$. Hence,

$$\big((t_1^{(1)}, t_2^{(1)}), (t_1^{(2)}, t_2^{(2)}), (t_1^{(3)}, t_2^{(3)})\big) = \big((t, 1-t), (1,0), (0,1)\big) \text{ for } 0 \le t \le 1.$$

Therefore, $\text{Norm}(T) = \Big\{\Big(\pm(t,-(1-t)), \pm(1,0), \pm(0,1)\Big), \Big(\pm(t,-(1-t)), \pm(0,1), \pm(1,0)\Big), \Big(\pm(t,-(1-t)), \pm(1,0), \pm(1,0)\Big), \Big(\pm(t,1-t), \pm(0,1), \pm(0,1)\Big), \Big(\pm(1,0), \pm(1,0), \pm(t,-(1-t))\Big), \Big(\pm(0,1), \pm(0,1), \pm(t,-(1-t))\Big), \Big(\pm(1,0), \pm(0,1), \pm(t,1-t)\Big), \Big(\pm(0,1), \pm(1,0), \pm(t,1-t)\Big), \Big(\pm(1,0), \pm(t,1-t), \pm(1,0)\Big), \Big(\pm(1,0), \pm(t,-(1-t)), \pm(0,1)\Big), \Big(\pm(0,1), \pm(t,1-t), \pm(0,1)\Big), \Big(\pm(0,1), \pm(t,-(1-t)), \pm(1,0)\Big) : 0 \le t, s \le 1\Big\}$.

The proof of Case 2 follows from Proposition D. This completes the proof. $\square$

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