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PSEUDOSTARLIKE AND PSEUDOCONVEX IN A DIRECTION MULTIPLE DIRICHLET SERIES

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The article introduces the concepts of pseudostarlikeness and pseudoconvexity in the direction of absolutely converges in $\Pi_0 = \{s \in \mathbb{C}^p \colon \operatorname{Re} s < 0\}$, $p \in \mathbb{N}$, the multiple Dirichlet series of the form

$$F(s) = e^{(h,s)} + \sum\nolimits_{\|(n)\|=\|(n^0)\|}^{+\infty} f_{(n)} \exp\{(\lambda_{(n)}, s)\}, \quad s = (s_1, ..., s_p) \in \mathbb{C}^p, \quad p \geq 1,$$

where $\lambda_{(n^0)} > h$, $\operatorname{Re} s < 0 \iff (\operatorname{Re} s_1 < 0, ..., \operatorname{Re} s_p < 0)$, $h = (h_1, ..., h_p) \in \mathbb{R}^p_+$, $(n) = (n_1, ..., n_p) \in \mathbb{N}^p$, $(n^0) = (n^0_1, ..., n^0_p) \in \mathbb{N}^p$, $\|(n)\| = n_1 + ... + n_p$ and the sequences $\lambda_{(n)} = (\lambda^{(1)}_{n_1}, ..., \lambda^{(p)}_{n_p})$ are such that $0 < h_j < \lambda^{(j)}_1 < \lambda^{(j)}_k < \lambda^{(j)}_{k+1} \uparrow +\infty$ as $k \to +\infty$ for every $j \in \{1, ..., p\}$, and $(a, c) = a_1c_1 + ... + a_pc_p$ for $a = (a_1, ..., a_p)$ and $c = (c_1, ..., c_p)$. We say that $a > c$ if $a_j \geq c_j$ for all $1 \leq j \leq p$ and there exists at least one j such that $a_j > c_j$. Let $\mathbf{b} = (b_1, ..., b_p)$ and $\partial_{\mathbf{b}} F(s) = \sum\limits_{j=1}^{p} b_j \dfrac{\partial F(s)}{\partial s_j}$ be the derivative of F in the direction $\mathbf{b}$. In this paper, in particular, the following assertions were obtained: 1) If $\mathbf{b} > 0$ and $\sum\limits_{\|(n)\|=k_0}^{+\infty} (\lambda_{(n)}, \mathbf{b})|f_{(n)}| \leq (h, \mathbf{b})$ then $\partial_{\mathbf{b}} F(s) \neq 0$ in $\Pi_0 := \{s \colon \operatorname{Re} s < 0\}$, i.e. F is conformal in Π_0 in the direction $\mathbf{b}$ (Proposition 1). 2) We say that function F is pseudostarlike of the order $\alpha \in [0, (h, \mathbf{b}))$ and the type $\beta > 0$ in the direction $\mathbf{b}$ if $\left|\frac{\partial_{\mathbf{b}} F(s)}{F(s)} - (h, \mathbf{b})\right| < \beta \left|\frac{\partial_{\mathbf{b}} F(s)}{F(s)} - (2\alpha - (h, \mathbf{b}))\right|$, $\quad s \in \Pi_0$. Let $0 \leq \alpha < (h, \mathbf{b})$ and $\beta > 0$. In order that the function F is pseudostarlike of the order α and the type β in the direction $\mathbf{b} > 0$, it is sufficient and in the case, when all $f_{(n)} \leq 0$, it is necessary that $\sum\limits_{\|(n)\|=k_0}^{+\infty} \{((1+\beta)\lambda_{(n)} - (1-\beta)h, \mathbf{b}) - 2\beta\alpha\}|f_{(n)}| \leq 2\beta((h, \mathbf{b}) - \alpha)$ (Theorem 1).

1. Introduction.

Let S be a class of functions

$$f(z) = z + \sum_{n=2}^{+\infty} f_n z^n \tag{1}$$

analytic univalent in $\mathbb{D} = \{z : |z| < 1\}$, i.e. conformal at every point $z \in \mathbb{D}$, that is $f'(z) \neq 0$, and injective on $\mathbb{D}$. Function $f \in S$ is said to be *starlike* if $f(\mathbb{D})$ is starlike domain concerning of the origin. It is well known [1, p.202] that the condition $\operatorname{Re}\{zf'(z)/f(z)\} > 0$ $(z \in \mathbb{D})$

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is necessary and sufficient for the starlikeness of f. A.W. Goodman [2] (see also [3, p.9]) proved that if $\sum_{n=2}^{+\infty} n|f_n| \le 1$ then function f of form (1) is starlike.

The concept of the starlikeness of function (1) got the series of generalizations. I.S. Jack [4] studied starlike functions of order $\alpha \in [0, 1)$, i.e. such functions (1), for which

$$\mathrm{Re}\,\{zf'(z)/f(z)\} > \alpha \ (z \in \mathbb{D}).$$

It is proved [4], [3, p.13] that if

$$\sum_{n=2}^{+\infty}(n-\alpha)|f_n| \le 1-\alpha$$

then function of form (1) is starlike function of order α. V.P. Gupta [5] introduced the concept of *starlike function of order* $\alpha \in [0, 1)$ *and type* $\beta \in (0, 1]$. A function (1) is so named for that

$$\Big|\frac{zf'(z)}{f(z)} - 1\Big| < \beta\Big|\frac{zf'(z)}{f(z)} + 1 - 2\alpha\Big|.$$

It is proved [5] that if

$$\sum_{n=2}^{+\infty}\{(1+\beta)n - \beta(2\alpha-1) - 1)\}|f_n| \le 2\beta(1-\alpha)$$

then function (1) is starlike function of order α and type β.

For $f \in S$, following A.W. Goodman [6] and S. Ruscheweyh [7], its neighborhood is called a set

$$N_\delta(f) = \Big\{g(z) = z + \sum_{k=2}^{+\infty} g_k z^k \in S: \sum_{k=2}^{+\infty} k|g_k - f_k| \le \delta\Big\}. \tag{2}$$

It is known [6] that if $f(z) = z$ and $g \in N_1(f)$ then g is starlike function. The neighborhoods of various classes of analytical in $\mathbb{D}$ functions were studied by many authors (we indicate here only on articles [8]–[14]).

For power series $f_j(z) = \sum\limits_{k=0}^{+\infty} f_{k,j} z^k$ $(j = 1, 2)$ the series $(f_1 * f_2)(z) = \sum\limits_{k=0}^{+\infty} f_{k,1} f_{k,2} z^k$ is called the Hadamard composition (product) [15]. Obtained by J.Hadamard properties of this composition find the applications [16]–[17] in the theory of the analytic continuation of the functions represented by power series. We remark also that singular points of the Hadamard composition are investigated in the article [18]. L.Zalzman [19] studied Hadamard compositions of univalent functions of the form (1). For the functions $f_j(z) = 1/z + \sum\limits_{k=1}^{+\infty} f_{k,j} z^k$ $(j = 1, 2)$ M.L. Mogra [20] defined Hadamard composition as $(f_1 * f_2)(z) = 1/z + \sum\limits_{k=1}^{+\infty} f_{k,1} f_{k,2} z^k$ and proved, for example, that if the functions f_j are *meromorphically starlike of the order* $\alpha_j \in [0, 1)$ and $f_{k,j} \ge 0$ for all $k \ge 1$ then $f_1 * f_2$ is meromorphically starlike of the order $\alpha = \max\{\alpha_1, \alpha_2\}$. Hadamard compositions of analytic and meromorphic functions in $\mathbb{D}$ studied also by J.H. Choi, Y.C. Kim and S. Owa [21], M.K. Aouf and H. Silverman [22], J.Liu and R. Srivastava [23], S. Ruscheweyh [7] and many other mathematicians.

Let $h \ge 1$ and (λ_k) be an increasing to $+\infty$ sequence of positive numbers $(\lambda_1 > h)$. Absolutely convergent in the half-plane $\Pi_0 = \{s: \mathrm{Re}\, s < 0\}$ Dirichlet series of the form

$$F(s) = e^{sh} + \sum_{k=1}^{+\infty} f_k \exp\{s\lambda_k\}, \quad s = \sigma + it, \tag{3}$$

are direct generalizations of the functions $f\in S$.

It is known [24], [3, p.135] that each function (3) is non-univalent in Π_0, but there exist conformal in Π_0 functions (3), and if

$$\sum_{k=2}^{+\infty}\lambda_k|f_k|\le h, \tag{4}$$

then function F of form (3) is conformal in Π_0. A conformal function F in Π_0 is said to be pseudostarlike if $\mathrm{Re}\{F'(s)/F(s)\}>0\,(s\in\Pi_0)$. In [24] (see also [3, p.139]) it is proved that if (4) holds then function F of form (3) is pseudostarlike.

A conformal function (3) in Π_0 is said to be *pseudostarlike of the order* α if

$$\mathrm{Re}\{F'(s)/F(s)\}>\alpha\in[0,\,1),\quad s\in\Pi_0. \tag{5}$$

Since the inequality $|w-h|<|w-(2\alpha-h)|$ holds if and only if $\mathrm{Re}\,w>\alpha$, function (3) is pseudostarlike of the order α if and only if

$$\Big|\frac{F'(s)}{F(s)}-h\Big|<\Big|\frac{F'(s)}{F(s)}-(2\alpha-h)\Big|,\quad s\in\Pi_0. \tag{6}$$

In view of (6), as in [25], we call conformal function (3) in Π_0 *pseudostarlike of the order* $\alpha\in[0,\,1)$ *and the type* $\beta\in(0,\,1]$ if

$$\Big|\frac{F'(s)}{F(s)}-h\Big|<\beta\Big|\frac{F'(s)}{F(s)}-(2\alpha-h)\Big|,\quad s\in\Pi_0. \tag{7}$$

In [25] it is proved that if

$$\sum_{k=1}^{+\infty}\{(1+\beta)\lambda_k-2\beta\alpha-h(1-\beta)\}|f_k|\le 2\beta(h-\alpha)$$

then (3) is pseudostarlike of the order α and the type β.

If in the definition of the pseudostarlikeness instead of F'/F to put F''/F' then will get the definition of *the pseudoconvexity*.

S.M. Shah [27] indicated conditions on real parameters γ_0, γ_1, γ_2 of the differential equation $z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}+(\gamma_0z^2+\gamma_1z+\gamma_2)w=0$, under which there exists an entire transcendental solution (1) such that f and all its derivatives are close-to-convex in $\mathbb{D}$. The convexity of solutions of the Shah equation has been studied in [28 - 29]. Substituting $z=e^s$ we obtain the differential equation $\frac{d^2w}{ds^2}+(\gamma_0e^{2hs}+\gamma_1e^{hs}+\gamma_2)w=0$ with $h=1$. The pseudoconvexity and pseudostarlikeness of solutions of the last equation has been studied in [3, p.147–153].

Here we will get similar results for multiple Dirichlet series.

2. Pseudostarlikeness and pseudoconvexity Let $p\in\mathbb{N}$, $h=(h_1,...,h_p)\in\mathbb{R}^p_+$, $(n)=(n_1,...,n_p)\in\mathbb{N}^p$, $(n^0)=(n^0_1,...,n^0_p)\in\mathbb{N}^p$, $\|(n)\|=n_1+...+n_p$ and the sequences $\lambda_{(n)}=(\lambda^{(1)}_{n_1},...,\lambda^{(p)}_{n_p})$ are such that $0<\lambda^{(j)}_1<\lambda^{(j)}_k<\lambda^{(j)}_{k+1}\uparrow+\infty$ as $k\to\infty$ for every $j\in\{1,...,p\}$. Also let $s=(s_1,...,s_p)\in\mathbb{C}^p$, $s_j=\sigma_j+it_j$, $\sigma=(\sigma_1,...,\sigma_p)$, and for $a=(a_1,...,a_p)$ and $c=(c_1,...,c_p)$ let $(a,c)=a_1c_1+...+a_pc_p$. We say that $a>c$ if $a_j\ge c_j$ for all $1\le j\le p$ and there exists at least one j such that $a_j>c_j$.

Suppose that the multiple Dirichlet series

$$F(s) = e^{(s,h)} + \sum_{\|(n)\|=\|(n^0)\|}^{+\infty} f_{(n)} \exp\{(\lambda_{(n)}, s)\}, \quad \lambda_{(n^0)} > h, \tag{8}$$

absolutely converges in $\Pi_0 = \{s : \operatorname{Re} s < 0\}$, where $\operatorname{Re} s < 0 \iff (\operatorname{Re} s_1 < 0, ..., \operatorname{Re} s_p < 0)$. We will use the notation $k_0 := \|(n^0)\|$ everywhere.

For the definition of the pseudostarlikeness of function (8) can be used either a partial derivative of one variable or the derivative in joint variables [30] or the derivative in the direction [31]. Here we will stop on the derivative in the direction.

Let $\mathbf{b} = (b_1, ..., b_p)$ and $\partial_{\mathbf{b}} F(s) = \sum_{j=1}^{p} b_j \frac{\partial F(s)}{\partial s_j}$ be the derivative of F in the direction $\mathbf{b}$.

Proposition 1. *If* $\mathbf{b} > 0$ *and*

$$\sum_{\|(n)\|=k_0}^{+\infty} (\lambda_{(n)}, \mathbf{b})|f_{(n)}| \le (h, \mathbf{b}) \tag{9}$$

then $\partial_{\mathbf{b}} F(s) \ne 0$ *in* Π_0, *i.e.* F *is conformal in* Π_0 *in the direction* $\mathbf{b}$.

Indeed,

$$|\partial_{\mathbf{b}} F(s)| =$$

$$= \Big|(h_1 b_1 + ... + h_p b_p) e^{s_1 h_1 + ... + s_p h_p} + \sum_{\|(n)\|=k_0}^{+\infty} (b_1 \lambda_{n_1}^{(1)} + ... + b_p \lambda_{n_p}^{(p)}) f_{(n)} \exp\{(\lambda_{(n)}, s)\}\Big| \ge$$

$$\ge \Big|(h, \mathbf{b}) e^{(s,h)}\Big| - \Big|\sum_{\|(n)\|=k_0}^{+\infty} (\lambda_{(n)}, \mathbf{b})) f_{(n)} \exp\{(\lambda_{(n)}, s)\}\Big| =$$

$$= \Big|(h, \mathbf{b}) e^{(s,h)}\Big| \Big(1 - \Big|\sum_{\|(n)\|=k_0}^{+\infty} \frac{(\lambda_{(n)}, \mathbf{b})}{(h, \mathbf{b})} f_{(n)} \exp\{(\lambda_{(n)} - h, s)\}\Big|\Big) \ge$$

$$\ge \Big|(h, \mathbf{b}) e^{(\sigma,h)}\Big| \Big(1 - \Big|\sum_{\|(n)\|=k_0}^{+\infty} \frac{(\lambda_{(n)}, \mathbf{b})}{(h, \mathbf{b})} |f_{(n)}| \exp\{(\lambda_{(n)} - h, \sigma)\}\Big|\Big) >$$

$$> \Big|(h, \mathbf{b}) e^{(\sigma,h)}\Big| \Big(1 - \Big|\sum_{\|(n)\|=k_0}^{+\infty} \frac{(\lambda_{(n)}, \mathbf{b})}{(h, \mathbf{b})} |f_{(n)}|\Big|\Big) \ge 0,$$

i.e. F is conformal in Π_0 in the direction $\mathbf{b}$.

We say that function (8) is *pseudostarlike of the order* $\alpha \in [0, (h, \mathbf{b}))$ *and the type* $\beta > 0$ in the direction $\mathbf{b}$ if

$$\Big|\frac{\partial_{\mathbf{b}} F(s)}{F(s)} - (h, \mathbf{b})\Big| < \beta \Big|\frac{\partial_{\mathbf{b}} F(s)}{F(s)} - (2\alpha - (h, \mathbf{b}))\Big|, \quad s \in \Pi_0. \tag{10}$$

Theorem 1. *Let* $0 \le \alpha < (h, \mathbf{b})$ *and* $\beta > 0$. *In order that the function (1) is pseudostarlike of the order* α *and the type* β *in the direction* $\mathbf{b} > 0$, *it is sufficient and in the case, when all* $f_{(n)} \le 0$, *it is necessary that*

$$\sum_{\|(n)\|=k_0}^{+\infty} \{((1+\beta)\lambda_{(n)} - (1-\beta)h, \mathbf{b}) - 2\beta\alpha\} |f_{(n)}| \le 2\beta((h, \mathbf{b}) - \alpha) \tag{11}$$

Proof. Clearly, (10) holds if and only if

$$\Big|\partial_{\mathbf{b}}F(s)-(h,\mathbf{b})F(s)\Big|-\beta\Big|\partial_{\mathbf{b}}F(s)-(2\alpha-(h,\mathbf{b}))F(s)\Big|<0,\quad s\in\Pi_0. \tag{12}$$

On the other hand,

$$\Big|\partial_{\mathbf{b}}F(s)-(h,\mathbf{b})F(s)\Big|-\beta\Big|\partial_{\mathbf{b}}F(s)-(2\alpha-(h,\mathbf{b}))F(s)\Big|=$$

$$=\Big|(h,\mathbf{b})e^{(h,s)}+\sum_{\|(n)\|=k_0}^{+\infty}(\lambda_{(n)},\mathbf{b})f_{(n)}\exp\{(\lambda_{(n)},s)\}-$$

$$-(h,\mathbf{b})e^{(h,s)}-(h,\mathbf{b})\sum_{\|(n)\|=k_0}^{+\infty}f_{(n)}\exp\{(\lambda_{(n)},s)\}\Big|-$$

$$-\beta\Big|(h,\mathbf{b})e^{(h,s)}+\sum_{\|(n)\|=k_0}^{+\infty}(\lambda_{(n)},\mathbf{b})f_{(n)}\exp\{(\lambda_{(n)},s)\}-$$

$$-(2\alpha-(h,\mathbf{b}))e^{(h,s)}-(2\alpha-(h,\mathbf{b}))\sum_{\|(n)\|=k_0}^{+\infty}f_{(n)}\exp\{(\lambda_{(n)},s)\}\Big|=$$

$$=\Big|\sum_{\|(n)\|=k_0}^{+\infty}(\lambda_{(n)}-h,\mathbf{b})f_{(n)}\exp\{(\lambda_{(n)},s)\}\Big|-$$

$$-\beta\Big|2((h,\mathbf{b})-\alpha)e^{(h,s)}+\sum_{\|(n)\|=k_0}^{+\infty}((\lambda_{(n)}+h,\mathbf{b})-2\alpha)f_{(n)}\exp\{(\lambda_{(n)},s)\}\Big|$$

Since $-|a+b|\le-|a|+|b|$ and $\sigma_j<0$, hence by condition $(h,\mathbf{b})-\alpha>0$ we get

$$\Big|\partial_{\mathbf{b}}F(s)-(h,\mathbf{b})F(s)\Big|-\beta\Big|\partial_{\mathbf{b}}F(s)-(2\alpha-(h,\mathbf{b}))F(s)\Big|\le$$

$$\le\Big|\sum_{\|(n)\|=k_0}^{+\infty}(\lambda_{(n)}-h,\mathbf{b})f_{(n)}\exp\{(\lambda_{(n)},s)\}\Big|-\Big|2\beta((h,\mathbf{b})-\alpha)e^{(h,s)}\Big|+$$

$$+\Big|\sum_{\|(n)\|=k_0}^{+\infty}\beta((\lambda_{(n)}+h,\mathbf{b})-2\alpha)f_{(n)}\exp\{(\lambda_{(n)},s)\}\Big|\le$$

$$\le\sum_{\|(n)\|=k_0}^{+\infty}(\lambda_{(n)}-h,\mathbf{b})|f_{(n)}|\exp\{(\lambda_{(n)},\sigma)\}-2\beta((h,\mathbf{b})-\alpha)e^{(h,\sigma)}+$$

$$+\sum_{\|(n)\|=k_0}^{+\infty}\beta((\lambda_{(n)}+h,\mathbf{b})-2\alpha)|f_{(n)}|\exp\{(\lambda_{(n)},\sigma)\}=$$

$$=\sum_{\|(n)\|=k_0}^{+\infty}\Big(((1+\beta)\lambda_{(n)}-(1-\beta)h,\mathbf{b}\Big)-2\beta\alpha\Big)|f_{(n)}|\exp\{(\lambda_{(n)},\sigma)\}-$$

$$-2\beta((h,\mathbf{b})-\alpha)e^{(h,\sigma)}=$$

$$= e^{(h,\sigma)}\Big(\sum_{\|(n)\|=k_0}^{+\infty}\Big(((1+\beta)\lambda_{(n)}-(1-\beta)h,\mathbf{b})-2\beta\alpha\Big)|f_{(n)}|\exp\{(\lambda_{(n)}-h,\sigma)\}-$$

$$-2\beta((h,\mathbf{b})-\alpha)\Big) <$$

$$< e^{(h,\sigma)}\Big(\sum_{\|(n)\|=k_0}^{+\infty}\{((1+\beta)\lambda_{(n)}-(1-\beta)h,\mathbf{b})-2\beta\alpha\}|f_{(n)}|-2\beta((h,\mathbf{b})-\alpha)\Big)\le 0$$

if (11) holds. The sufficiency of (11) is proved

Now suppose that (12) holds and all $f_{(n)} \le 0$. Then

$$\left|\frac{-\sum\limits_{\|(n)\|=k_0}^{+\infty}(\lambda_{(n)}-h,\mathbf{b})f_{(n)}\exp\{(\lambda_{(n)},s)\}}{2((h,\mathbf{b})-\alpha)e^{(h,s)}-\sum\limits_{\|(n)\|=k_0}^{+\infty}((\lambda_{(n)}+h,\mathbf{b})-2\alpha)f_{(n)}\exp\{(\lambda_{(n)},s)}\right| =$$

$$= \left|\frac{\partial_{\mathbf{b}}F(s)-(h,\mathbf{b})F(s)}{\partial_{\mathbf{b}}F(s)-(2\alpha-(h,\mathbf{b}))F(s)}\right| < \beta.$$

Therefore,

$$\mathrm{Re}\frac{\sum\limits_{\|(n)\|=k_0}^{+\infty}(\lambda_{(n)}-h,\mathbf{b})f_{(n)}\exp\{(\lambda_{(n)},s)\}}{2((h,\mathbf{b})-\alpha)e^{(h,s)}-\sum\limits_{\|(n)\|=k_0}^{+\infty}((\lambda_{(n)}+h,\mathbf{b})-2\alpha)f_{(n)}\exp\{(\lambda_{(n)},s)} < \beta,$$

and in particular for $\sigma < 0$ we have

$$\mathrm{Re}\frac{\sum\limits_{\|(n)\|=k_0}^{+\infty}(\lambda_{(n)}-h,\mathbf{b})f_{(n)}\exp\{(\lambda_{(n)},\sigma)\}}{2((h,\mathbf{b})-\alpha)e^{(h,\sigma)}-\sum\limits_{\|(n)\|=p}^{+\infty}((\lambda_{(n)}+h,\mathbf{b})-2\alpha)f_{(n)}\exp\{(\lambda_{(n)},\sigma)} < \beta,$$

Letting $\sigma \to 0$, we obtain

$$\frac{\sum\limits_{\|(n)\|=k_0}^{+\infty}(\lambda_{(n)}-h,\mathbf{b})f_{(n)}}{2((h,\mathbf{b})-\alpha)-\sum\limits_{\|(n)\|=k_0}^{+\infty}((\lambda_{(n)}+h,\mathbf{b})-2\alpha)f_{(n)}} \le \beta,$$

whence (11) follows. The proof of Theorem 1 is complete d. □

Dirichlet series (8) is said to be pseudoconvex of the order α and the type β in the direction $\mathbf{b}$ if

$$\left|\frac{\partial^2_{\mathbf{b}}F(z)}{\partial_{\mathbf{b}}F(s)}-(h,\mathbf{b})\right| < \beta\left|\frac{\partial^2_{\mathbf{b}}F(z)}{\partial_{\mathbf{b}}F(s)}-(2\alpha-(h,\mathbf{b}))\right|,\quad s\in\Pi_0. \tag{13}$$

Clearly, the function F is pseudoconvex of the order α and the type β in the direction $\mathbf{b}$ if and only if the function $\partial_{\mathbf{b}}F(s)$ is pseudostarlike of the order α and the type β in the direction $\mathbf{b}$ and, thus, if and only if the function $\frac{\partial_{\mathbf{b}}F(s)}{(h,\mathbf{b})}$ is pseudostarlike of the order α and the type β in the direction $\mathbf{b}$. Since

$$\frac{\partial_{\mathbf{b}}F(s)}{(h,\mathbf{b})} = e^{(s,h)} + \sum_{\|(n)\|=k_0}^{+\infty} f^*_{(n)} \exp\{(\lambda_{(n)}, s)\}, \quad f^*_{(n)} = \frac{(\lambda_{(n)},\mathbf{b})}{(h,\mathbf{b})} f_{(n)},$$

by Theorem 1 the condition

$$\sum_{\|(n)\|=k_0}^{+\infty} \{((1+\beta)\lambda_{(n)} - (1-\beta)h,\mathbf{b}) - 2\beta\alpha\} f^*_{(n)} \le 2\beta\ ((h,\mathbf{b}) - \alpha)$$

is sufficient and in the case, when all $f_{(n)} \le 0$, is necessary in order that $\frac{\partial_{\mathbf{b}}F(s)}{(h,\mathbf{b})}$ is pseudostarlike of the order α and the type β in the direction $\mathbf{b}$. Therefore, the following statement is true.

Proposition 2. *Let $0 \le \alpha < (h,\mathbf{b})$ and $\beta > 0$. In order that the function (8) is pseudoconvex of the order α and the type β in the direction $\mathbf{b} > 0$, it is sufficient and in the case, when all $f_{(n)} \le 0$, it is necessary that*

$$\sum_{\|(n)\|=k_0}^{+\infty} (\lambda_{(n)},\mathbf{b})\{((1+\beta)\lambda_{(n)} - (1-\beta)h,\mathbf{b}) - 2\beta\alpha\}|f_{(n)}| \le 2(h,\mathbf{b})\beta((h,\mathbf{b}) - \alpha). \tag{14}$$

Recall that the function (8) is called *pseudostarlike of the order* α if $\mathrm{Re}\frac{\partial_{\mathbf{b}}F(s)}{F(s)} > \alpha$, that is (10) holds with $\beta = 1$, and called *pseudostarlike*, if $\mathrm{Re}\frac{\partial_{\mathbf{b}}F(s)}{F(s)} > 0$, that is (10) holds with $\beta = 1$ and $\alpha = 0$. Therefore, Theorem 1 and Proposition 2 imply the following statements.

Corollary 1. *Let $0 \le \alpha < (h,\mathbf{b})$. In order that the function (8) is pseudostarlike of the order α in the direction $\mathbf{b} > 0$, it is sufficient and in the case, when all $f_{(n)} \le 0$, it is necessary that*

$$\sum_{\|(n)\|=k_0}^{+\infty} \{(\lambda_{(n)},\mathbf{b}) - \alpha\}|f_{(n)}| \le (h,\mathbf{b}) - \alpha. \tag{15}$$

In order that the function (8) is pseudoconvex of the order α in the direction $\mathbf{b} > 0$, it is sufficient and in the case, when all $f_{(n)} \le 0$, it is necessary that

$$\sum_{\|(n)\|=k_0}^{+\infty} (\lambda_{(n)},\mathbf{b})\{(\lambda_{(n)},\mathbf{b}) - \alpha\}|f_{(n)}| \le (h,\mathbf{b})((h,\mathbf{b}) - \alpha).$$

Corollary 2. *In order that the function (8) is pseudostarlike in the direction $\mathbf{b} > 0$, it is sufficient and in the case, when all $f_{(n)} \le 0$, it is necessary that (9) holds. In order that the function (8) is pseudoconvex in the direction $\mathbf{b} \ge 0$, it is sufficient and in the case, when all $f_{(n)} \le 0$, it is necessary that $\sum\limits_{\|(n)\|=k_0}^{+\infty} (\lambda_{(n)},\mathbf{b})^2|f_{(n)}| \le (h,\mathbf{b})^2$.*

3. Neighborhoods of multiple Dirichlet series. Here the class of series (8) absolutely convergent in Π_0 we denote by D and we say that $F \in D^*$ if $F \in D$ and all $f_{(n)} \le 0$. By $PSD_{\alpha,\beta}(\mathbf{b})$ we denote a class of pseudostarlike functions (8) of the order α and the type β in the direction $\mathbf{b}> 0$ and by $PCD_{\alpha,\beta}(\mathbf{b})$ we denote a class of pseudoconvex functions (8) of the order α and the type β in the direction $\mathbf{b}> 0$.

For $j > 0$ and $\delta > 0$ we define the *neighborhood* of $F \in D$ in the direction $\mathbf{b}$ as follows

$$O_{j,\delta}(F) := \Big\{G(s) = e^{(s,h)} + \sum_{\|(n)\|=k_0}^{+\infty} g_{(n)}e^{(\lambda_{(n)},s)} \in D : \sum_{\|(n)\|=k_0}^{+\infty} (\lambda_{(n)}, \mathbf{b})^j |g_{(n)} - f_{(n)}| \le \delta\Big\}. \quad (16)$$

Similarly, for $F \in D^*$

$$O^*_{j,\delta}(F) := \Big\{G(s) = e^{(s,h)} + \sum_{\|(n)\|=k_0}^{+\infty} g_{(n)}e^{(\lambda_{(n)},s)} \in D^* : \sum_{\|(n)\|=k_0}^{+\infty} (\lambda_{(n)}, \mathbf{b})^j |g_{(n)} - f_{(n)}| \le \delta\Big\}. \quad (17)$$

For $F \in D$ here we will establish a connection between classes $PSD_{\alpha,\beta}(\mathbf{b})$, $PCD_{\alpha,\beta}(\mathbf{b})$ and $O_{j,\delta}(F)$, $O^*_{j,\delta}(F)$.

We need the following lemma.

Lemma 1. *Let $F \in D$, $\mathbf{b} > 0$. Then $G \in O_{2,(h,\mathbf{b})\delta}(F)$ if and only if* $\dfrac{\partial_{\mathbf{b}} G}{(h, \mathbf{b})} \in O_{1,\delta}\Big(\dfrac{\partial_{\mathbf{b}} F}{(h, \mathbf{b})}\Big)$.

Indeed,

$$\frac{\partial_{\mathbf{b}} F(s)}{(h, \mathbf{b})} = e^{(h,s)} + \sum_{\|(n)\|=k_0}^{+\infty} \frac{(\lambda_{(n)}, \mathbf{b})}{(h, \mathbf{b})} f_{(n)}e^{(\lambda_{(n)},s)} \in D.$$

Therefore,

$$\frac{\partial_{\mathbf{b}} G}{(h, \mathbf{b})} \in O_{1,\delta}\Big(\frac{\partial_{\mathbf{b}} F}{(h, \mathbf{b})}\Big)$$

if and only if

$$\sum_{\|(n)\|=k_0}^{+\infty} (\lambda_{(n)}, \mathbf{b})\Big|\frac{(\lambda_{(n)}, \mathbf{b})}{(h, \mathbf{b})} f_{(n)} - \frac{(\lambda_{(n)}, \mathbf{b})}{(h, \mathbf{b})} g_{(n)}\Big| \le \delta,$$

i.e. $G \in O_{2,(h,\mathbf{b})\delta}(F)$.

At first we consider the case when $F(s) = E(s) := e^{(h,s)}$ and we prove such theorem.

Theorem 2. *Let $\mathbf{b} > 0$. The following relations are correct for the function $E(s) = e^{(h,s)}$: $O_{1,(h,\mathbf{b})}(E) \subset PSD_{0,1}(\mathbf{b})$, $O^*_{1,(h,\mathbf{b})}(E) = PSD_{0,1} \bigcap D^*(\mathbf{b})$, $O_{2,(h,\mathbf{b})^2}(E) \subset PCD_{0,1}(\mathbf{b})$ and $O^*_{2,(h,\mathbf{b})^2}(E) = PCD_{0,1} \bigcap D^*(\mathbf{b})$.*

Proof. If $G \in O_{1,(h,\mathbf{b})}(E)$ then $G \in D$ and $\sum\limits_{\|(n)\|=p}^{+\infty} (\lambda_{(n)}, \mathbf{b})|g_{(n)}| \le (h, \mathbf{b})$. Therefore,

$$\Big|\frac{\partial_{\mathbf{b}} G(s)}{(h, \mathbf{b})} - G(s)\Big| = \Big| \sum_{\|(n)\|=k_0}^{+\infty} \frac{(\lambda_{(n)}, \mathbf{b})}{(h, \mathbf{b})} g_{(n)}e^{(\lambda_{(n)},s)} - \sum_{\|(n)\|=k_0}^{+\infty} g_{(n)}e^{(\lambda_{(n)},s)}\Big| =$$

$$= \Big| \sum_{\|(n)\|=k_0}^{+\infty} \Big(\frac{(\lambda_{(n)}, \mathbf{b})}{(h, \mathbf{b})} - 1\Big) g_{(n)}e^{(\lambda_{(n)},s)}\Big| \le \sum_{\|(n)\|=k_0}^{+\infty} \Big(\frac{(\lambda_{(n)}, \mathbf{b})}{(h, \mathbf{b})} - 1\Big) |g_{(n)}|e^{(\lambda_{(n)},\sigma)} =$$

$$= \sum_{\|(n)\|=k_0}^{+\infty} \frac{(\lambda_{(n)}, \mathbf{b})}{(h, \mathbf{b})} |g_{(n)}| e^{(\lambda_{(n)},\sigma)} - \sum_{\|(n)\|=k_0}^{+\infty} |g_{(n)}| e^{(\lambda_{(n)},\sigma)} \le$$

$$\le e^{(h,\sigma)} \sum_{\|(n)\|=k_0}^{+\infty} \frac{(\lambda_{(n)}, \mathbf{b})}{(h, \mathbf{b})} |g_{(n)}| - \sum_{\|(n)\|=k_0}^{+\infty} |g_{(n)}| e^{(\lambda_{(n)},\sigma)} \le e^{(h,\sigma)} - \sum_{\|(n)\|=k_0}^{+\infty} |g_{(n)}| e^{(\lambda_{(n)},\sigma)}.$$

On the other hand,

$$|G(s)| = \left| e^{(h,s)} + \sum_{\|(n)\|=k_0}^{+\infty} g_{(n)} e^{(\lambda_{(n)},s)} \right| \ge e^{(h,\sigma)} - \sum_{\|(n)\|=k_0}^{+\infty} |g_{(n)}| e^{(\lambda_{(n)},\sigma)}$$

and, thus, $\left| \frac{\partial_{\mathbf{b}} G(s)}{(h,\mathbf{b})} - G(s) \right| \le |G(s)|$, i.e. $\left| \frac{\partial_{\mathbf{b}} G(s)}{(h,\mathbf{b}) G(s)} - 1 \right| \le 1$ for all $s \in \Pi_0$. From hence it follows that $\mathrm{Re}\left\{ \frac{\partial_{\mathbf{b}} G(s)}{(h,\mathbf{b}) G(s)} \right\} > 0$, i.e. $G \in PSD_{0,1}(\mathbf{b})$ and $O_{1,(h,\mathbf{b})}(E) \subset PSD_{0,1}(\mathbf{b})$.

From above it follows that $O^*_{1,(h,\mathbf{b})}(E) \subset PSD_{0,1}(\mathbf{b})$. On the other hand, if $G \in D$ and $G \in PSD_{0,1}(\mathbf{b})$ then by Corollary 2 condition (9) holds, i.e. $G \in O^*_{1,(h,\mathbf{b})}(E)$. Thus, $PSD_{0,1}(\mathbf{b}) \bigcap D \subset O^*_{1,(h,\mathbf{b}1}(E)$ and $PSD_{0,1}(\mathbf{b}) \bigcap D^* = O^*_{1,(h,\mathbf{b})}(E)$.

Since $G \in PCD_{0,1}(\mathbf{b})$ if and only if $\frac{\partial_{\mathbf{b}} G}{(h,\mathbf{b})} \in PSD_{0,1}(\mathbf{b})$, and by Lemma 1 $G \in O_{2,(h,\mathbf{b})\delta}(E)$ if and only if $\frac{\partial_{\mathbf{b}} G}{(h,\mathbf{b})} \in O_{1,\delta}\left(\frac{\partial_{\mathbf{b}} E}{(h,\mathbf{b})} \right) = O_{1,\delta}(E)$, one can easily obtain the corresponding results for pseudoconvex functions. For example, if $G \in O_{2,(h,\mathbf{b})^2}(E)$ then $\frac{\partial_{\mathbf{b}} G}{(h,\mathbf{b})} \in O_{1,(h,\mathbf{b})}(E)$ and, thus, $\frac{\partial_{\mathbf{b}} G}{(h,\mathbf{b})} \in PSD_{0,1}(\mathbf{b})$ and $\partial_{\mathbf{b}} \in PSD_{0,1}(\mathbf{b})$. Therefore, $O_{2,(h,\mathbf{b})^2}(E) \subset PCD_{0,1}(\mathbf{b})$. The proof of Theorem 2 is complete d. □

Now we investigate the neighborhoods of a pseudostarlike function of the order α. The following theorem is true.

Theorem 3. *Let $\mathbf{b} > 0$, $0 \le \alpha_1 < \alpha < (h, \mathbf{b})$,*

$$\delta_1 = (\alpha - \alpha_1) \frac{(\lambda_{(n^0)} - h, \mathbf{b})}{(\lambda_{(n^0)}, \mathbf{b}) - \alpha},\ \delta_2 = \frac{(\lambda_{(n^0)}, \mathbf{b})((h, \mathbf{b}) - \alpha)}{(\lambda_{(n^0)}, \mathbf{b}) - \alpha} + \frac{(\lambda_{(n^0)}, \mathbf{b})((h, \mathbf{b}) - \alpha_1)}{(\lambda_{(n^0)}, \mathbf{b}) - \alpha_1}.$$

If $F \in D^* \bigcap PSD_{\alpha,1}(\mathbf{b})$ then $O^*_{1,\delta_1}(F) \subset PSD_{\alpha_1,1}(\mathbf{b})$ and $D^* \bigcap PSD_{\alpha_1,1}(\mathbf{b}) \subset O^*_{1,\delta_2}(F)$, $O^*_{2,(h,\mathbf{b})\delta_1}(F) \subset PSD_{\alpha_1,1}(\mathbf{b})$ and $D^* \bigcap PSD_{\alpha_1,1}(\mathbf{b}) \subset O^*_{1,(h,\mathbf{b})\delta_2}(F)$.

Proof. Since $F \in PSD_{\alpha,1}(\mathbf{b})$, by Corollary 1 condition (15) holds. Therefore, for $\alpha_1 < \alpha$ we get

$$\sum_{\|(n)\|=k_0}^{+\infty} \{(\lambda_{(n)}, \mathbf{b}) - \alpha_1\} |g_{(n)}| \le$$

$$\le \sum_{\|(n)\|=k_0}^{+\infty} \{(\lambda_{(n)}, \mathbf{b}) - \alpha_1\} |g_{(n)} - f_{(n)}| + \sum_{\|(n)\|=k_0}^{+\infty} \{(\lambda_{(n)}, \mathbf{b}) - \alpha_1\} |f_{(n)}| =$$

$$= \sum_{\|(n)\|=k_0}^{+\infty} \{(\lambda_{(n)}, \mathbf{b}) - \alpha_1\} |g_{(n)} - f_{(n)}| + \sum_{\|(n)\|=k_0}^{+\infty} \{(\lambda_{(n)}, \mathbf{b}) - \alpha\} |f_{(n)}| + (\alpha - \alpha_1) \sum_{\|(n)\|=k_0}^{+\infty} |f_{(n)}| \le$$

$$\le \delta_1 + (h, \mathbf{b}) - \alpha + (\alpha - \alpha_1) \sum_{\|(n)\|=k_0}^{+\infty} |f_{(n)}|.$$

But in view of (15)

$$((\lambda_{(n^0)},\mathbf{b})-\alpha)\sum_{\|(n)\|=k_0}^{+\infty}|f_{(n)}|\le\sum_{\|(n)\|=k_0}^{+\infty}\{(\lambda_{(n)},\mathbf{b})-\alpha\}|f_{(n)}|\le(h,\mathbf{b})-\alpha,$$

that is

$$\sum_{\|(n)\|=k_0}^{+\infty}|f_{(n)}|\le\frac{(h,\mathbf{b})-\alpha}{(\lambda_{(n^0)},\mathbf{b})-\alpha)}$$

and, thus,

$$\sum_{\|(n)\|=k_0}^{+\infty}\{(\lambda_{(n)},\mathbf{b})-\alpha_1\}|g_{(n)}|\le\delta_1+(h,\mathbf{b})-\alpha+(\alpha-\alpha_1)\frac{(h,\mathbf{b})-\alpha}{(\lambda_{(n^0)},\mathbf{b})-\alpha)}=(h,\mathbf{b})-\alpha_1,$$

i.e. by Corollary 1 the function G is pseudostarlike of the order α_1 and, thus,

$$O^*_{1,\delta_1}(F)\subset PSD_{\alpha_1,1}(\mathbf{b}).$$

Now suppose that $G\in D_0^*\bigcap PSD_{\alpha_1,1}(\mathbf{b})$. Then in view of (15) we have

$$\sum_{\|(n)\|=k_0}^{+\infty}(\lambda_{(n)},\mathbf{b})|g_{(n)}-f_{(n)}|\le\delta=\sum_{\|(n)\|=k_0}^{+\infty}\frac{(\lambda_{(n)},\mathbf{b})}{(\lambda_{(n)},\mathbf{b})-\alpha_1}((\lambda_{(n)},\mathbf{b})-\alpha_1)|g_{(n)}-f_{(n)}|\le$$

$$\le\frac{(\lambda_{(n^0)},\mathbf{b})}{(\lambda_{(n^0)},\mathbf{b})-\alpha_1}\sum_{\|(n)\|=k_0}^{+\infty}((\lambda_{(n)},\mathbf{b})-\alpha_1)|g_{(n)}-f_{(n)}|\le$$

$$\le\frac{(\lambda_{(n^0)},\mathbf{b})}{(\lambda_{(n^0)},\mathbf{b})-\alpha_1}\Big(\sum_{\|(n)\|=k_0}^{+\infty}((\lambda_{(n)},\mathbf{b})-\alpha_1)|g_{(n)}|+\sum_{\|(n)\|=k_0}^{+\infty}\frac{(\lambda_{(n)},\mathbf{b})-\alpha_1}{(\lambda_{(n)},\mathbf{b})-\alpha}((\lambda_{(n)},\mathbf{b})-\alpha)|f_{(n)}|\Big)$$

$$\le\frac{(\lambda_{(n^0)},\mathbf{b})}{(\lambda_{(n^0)},\mathbf{b})-\alpha_1}\Big((h,\mathbf{b})-\alpha_1+\frac{(\lambda_{(n^0)},\mathbf{b})-\alpha_1}{(\lambda_{(n^0)},\mathbf{b})-\alpha}((h,\mathbf{b})-\alpha)\Big)=\delta_2,$$

i.e. $G\in O^*_{1,\delta_2}(F)$ and, thus, $D^*\bigcap PSD_{\alpha_1,1}(\mathbf{b})\subset O^*_{1,\delta_2}(F)$.

Since $F\in PCD_{0,1}(\mathbf{b})$ if and only if $\frac{\partial_{\mathbf{b}}F}{(h,\mathbf{b})}\in PSD_{0,1}(\mathbf{b})$, and by Lemma 1 $G\in O_{2,(h,\mathbf{b})\delta}(F)$ if and only if $\frac{\partial_{\mathbf{b}}G}{(h,\mathbf{b})}\in O_{1,\delta}\Big(\frac{\partial_{\mathbf{b}}F}{(h,\mathbf{b})}\Big)$, one can easily obtain the corresponding results for pseudoconvex functions. For example, if $G\in O_{2,(h,\mathbf{b})\delta_1}(F)$ then $\frac{\partial_{\mathbf{b}}G}{(h,\mathbf{b})}\in O_{1,\delta_1}\Big(\frac{\partial_{\mathbf{b}}F}{(h,\mathbf{b})}\Big)$ and, thus, $\frac{\partial_{\mathbf{b}}G}{(h,\mathbf{b})}\in PSD_{\alpha_1,1}(\mathbf{b})$ and $G\in PSD_{\alpha_1,1}(\mathbf{b})$. Therefore, $O_{2,(h,\mathbf{b})\delta_1}(E)\subset PCD_{\alpha_1,1}(\mathbf{b})$.

The proof of Theorem 3 is complete d. □

Finally, we consider the generalized case when F is a pseudostarlike function of the order α and the type β. The following theorem is true.

Theorem 4. *Let* $\mathbf{b}>0$, $0\le\alpha<(h,\mathbf{b})$, $0<\beta<\beta_1\le1$,

$$A=\frac{((1+\beta_1)\lambda_{(n^0)}-(1-\beta_1)h,\mathbf{b})-2\alpha\beta_1}{((1+\beta)\lambda_{(n^0)}-(1-\beta)h,\mathbf{b})-2\alpha\beta},\quad\delta_1=\frac{\beta_1-A\beta}{1+\beta_1}((h,\mathbf{b})-\alpha)$$

$$\delta_2=\frac{2\beta_1(\lambda_{(n^0)},\mathbf{b})((h,\mathbf{b})-\alpha)}{((1+\beta_1)\lambda_{(p)}-(1-\beta_1)h,\mathbf{b})-2\alpha\beta_1}+\frac{2\beta(\lambda_{(n^0)},\mathbf{b})((h,\mathbf{b})-\alpha)}{((1+\beta)\lambda_{(p)}-(1-\beta)h,\mathbf{b})-2\alpha\beta}.$$

If $F\in D^*\bigcap PSD_{\alpha,\beta}(\mathbf{b})$ *then* $O^*_{1,\delta_1}(F)\subset PSD_{\alpha,\beta_1}(\mathbf{b})$ *and* $D^*\bigcap PSD_{\alpha,\beta_1}(\mathbf{b})\subset O^*_{1,\delta_2}(F)$, $O^*_{2,(h,\mathbf{b})\delta_1}(F)\subset PSD_{\alpha,\beta_1}(\mathbf{b})$ *and* $D^*\bigcap PSD_{\alpha,\beta_1}(\mathbf{b})\subset O^*_{2,(h,\mathbf{b})\delta_2}(F)$.

Proof. At first we remark that $\max\limits_{\|(n)\|\geq\|n^0\|}\frac{((1+\beta_1)\lambda_{(n)}-(1-\beta_1)h,\mathbf{b})-2\alpha\beta_1}{((1+\beta)\lambda_{(n)}-(1-\beta)h,\mathbf{b})-2\alpha\beta}=A$ and $\beta_1-A\beta>0$. For $0<\beta<\beta_1\leq 1$ we have

$$\sum_{\|n\|=k_0}^{+\infty}\{((1+\beta_1)\lambda_{(n)}-(1-\beta_1)h,\mathbf{b})-2\alpha\beta_1\}|g_{(n)}|\leq$$

$$\leq\sum_{\|n\|=k_0}^{+\infty}\{((1+\beta_1)\lambda_{(n)}-(1-\beta_1)h,\mathbf{b})-2\alpha\beta_1\}|g_{(n)}-f_{(n)}|+$$

$$+\sum_{\|n\|=k_0}^{+\infty}\{((1+\beta_1)\lambda_{(n)}-(1-\beta_1)h,\mathbf{b})-2\alpha\beta_1\}|f_{(n)}|. \tag{18}$$

If $G\in O^*_{1,\delta}(F)$ then

$$\sum_{\|n\|=k_0}^{+\infty}\{((1+\beta_1)\lambda_{(n)}-(1-\beta_1)h,\mathbf{b})-2\alpha\beta_1\}|g_{(n)}-f_{(n)}|\leq$$

$$=\sum_{\|n\|=k_0}^{+\infty}\{((1+\beta_1)\lambda_{(n)}+(\beta_1-1)h,\mathbf{b})\}|g_{(n)}-f_{(n)}|\leq$$

$$\leq\sum_{\|n\|=k_0}^{+\infty}\{((1+\beta_1)\lambda_{(n)}+(\beta_1-1)^+h,\mathbf{b})\}|g_{(n)}-f_{(n)}|=$$

$$=(1+\beta_1)\sum_{\|n\|=k_0}^{+\infty}\{(\lambda_{(n)}+\frac{(\beta_1-1)^+}{1+\beta_1}h,\mathbf{b})\}|g_{(n)}-f_{(n)}|\leq$$

$$\leq(1+\beta_1)\sum_{\|n\|=k_0}^{+\infty}\{(\lambda_{(n)}+h,\mathbf{b})\}|g_{(n)}-f_{(n)}|\leq$$

$$\leq 2(1+\beta_1)\sum_{\|n\|=k_0}^{+\infty}\{(\lambda_{(n)},\mathbf{b})\}|g_{(n)}-f_{(n)}|\leq 2(1+\beta_1)\delta$$

and, since $F\in D^*\bigcap PSD_{\alpha,\beta}(\mathbf{b})$, by Theorem 1

$$\sum_{\|n\|=k_0}^{+\infty}\{((1+\beta_1)\lambda_{(n)}-(1-\beta_1)h,\mathbf{b})-2\alpha\beta_1\}|f_{(n)}|=$$

$$=\sum_{\|n\|=k_0}^{+\infty}\frac{((1+\beta_1)\lambda_{(n)}-(1-\beta_1)h,\mathbf{b})-2\alpha\beta_1}{((1+\beta)\lambda_{(n)}-(1-\beta)h,\mathbf{b})-2\alpha\beta}\{((1+\beta)\lambda_{(n)}-(1-\beta)h,\mathbf{b})-2\alpha\beta\}|f_{(n)}|\leq$$

$$\leq A\sum_{\|n\|=k_0}^{+\infty}\{((1+\beta)\lambda_{(n)}-(1-\beta)h,\mathbf{b})-2\alpha\beta\}|f_{(n)}|\leq A2\beta((h,\mathbf{b})-\alpha).$$

Therefore, (18) implies

$$\sum_{\|n\|=k_0}^{+\infty}\{((1+\beta_1)\lambda_{(n)}-(1-\beta_1)h,\mathbf{b})-2\alpha\beta_1\}|g_{(n)}|\le 2(1+\beta_1)\delta+2A\beta((h,\mathbf{b})-\alpha)=2\beta_1((h,\mathbf{b})-\alpha),$$

i.e. by Theorem 1 $G\in PSD_{\alpha,\beta_1}(\mathbf{b})$.

Now suppose that $G\in D^*\bigcap PSD_{\alpha,\beta_1}(\mathbf{b})$. Then

$$\sum_{\|(n)\|=k_0}^{+\infty}(\lambda_{(n)},\mathbf{b})|g_{(n)}-f_{(n)}|=$$

$$=\sum_{\|(n)\|=k_0}^{+\infty}\frac{(\lambda_{(n)},\mathbf{b})\Big(((1+\beta_1)\lambda_{(n)}-(1-\beta_1)h,\mathbf{b})-2\alpha\beta_1\Big)}{((1+\beta_1)\lambda_{(n)}-(1-\beta_1)h,\mathbf{b})-2\alpha\beta_1}|g_{(n)}-f_{(n)}|\le$$

$$\le\frac{(\lambda_{(n^0)},\mathbf{b})}{((1+\beta_1)\lambda_{(n^0)}-(1-\beta_1)h,\mathbf{b})-2\alpha\beta_1}\times$$

$$\times\Big(\sum_{\|(n)\|=k_0}^{+\infty}\{((1+\beta_1)\lambda_{(n)}-(1-\beta_1)h,\mathbf{b})-2\alpha\beta_1\}|g_{(n)}|+$$

$$+\sum_{\|(n)\|=k_0}^{+\infty}\{((1+\beta_1)\lambda_{(n)}-(1-\beta_1)h,\mathbf{b})-2\alpha\beta_1\}|f_{(n)}|\Big)\le$$

$$\le\frac{(\lambda_{(n^0)},\mathbf{b})}{((1+\beta_1)\lambda_{(n^0)}-(1-\beta_1)h,\mathbf{b})-2\alpha\beta_1}\Big(2\beta_1((h,\mathbf{b}))-\alpha)+$$

$$+\sum_{\|(n)\|=k_0}^{+\infty}\frac{((1+\beta_1)\lambda_{(n)}-(1-\beta_1)h,\mathbf{b})-2\alpha\beta_1}{((1+\beta_1)\lambda_{(n)}-(1-\beta)h,\mathbf{b})-2\alpha\beta}\{((1+\beta_1)\lambda_{(n)}-(1-\beta)h,\mathbf{b})-2\alpha\beta\}|f_{(n)}|\Big)\le$$

$$\le\frac{(\lambda_{(n^0)},\mathbf{b})}{((1+\beta_1)\lambda_{(n^0)}-(1-\beta_1)h,\mathbf{b})-2\alpha\beta_1}\times$$

$$\times\Big(2\beta_1((h,\mathbf{b}))-\alpha)+\frac{((1+\beta_1)\lambda_{(n^0)}-(1-\beta_1)h,\mathbf{b})-2\alpha\beta_1}{((1+\beta_1)\lambda_{(n^0)}-(1-\beta)h,\mathbf{b})-2\alpha\beta}2\beta((h,\mathbf{b}))-\alpha)\Big)=$$

$$=\frac{2\beta_1(\lambda_{(n^0)},\mathbf{b})(((h,\mathbf{b}))-\alpha)}{((1+\beta_1)\lambda_{(n^0)}-(1-\beta_1)h,\mathbf{b})-2\alpha\beta_1}+\frac{2\beta(\lambda_{(n^0)},\mathbf{b})(((h,\mathbf{b}))-\alpha)}{((1+\beta)\lambda_{(n^0)}-(1-\beta)h,\mathbf{b})-2\alpha\beta}=\delta_2.$$

i.e. $G\in O^*_{1,\delta_2}(F)$ and $D^*\bigcap PSD_{\alpha,\beta_1}(\mathbf{b})\subset O^*_{1,\delta_2}(F)$.

Using the arguments applied in the proof of Theorem 3 we arrive at the corresponding results for pseudoconvex functions. The proof of Theorem 4 is complete d. □

4. Hadamard compositions of multiple Dirichlet series. For Dirichlet series $F_j(s)=e^{(s,h)}+\sum\limits_{\|(n)\|=k_0}^{+\infty}f_{(n),j}\exp\{(\lambda_{(n)},s)\}$ $(j=1,2)$ the Hadamard composition has the form

$$(F_1*F_2)(s)=e^{(s,h)}+\sum_{\|(n)\|=k_0}^{+\infty}f_{(n),1}f_{(n),2}\exp\{(\lambda_{(n)},s)\}. \tag{19}$$

Proposition 3. *Let* $\mathbf{b} > 0$. *If the functions* $F_j \in D^*$ *are pseudostarlike of the orders* $\alpha_j \in [0,\ (h,\mathbf{b}))$ *then Hadamard composition* $F_1 * F_2$ *is pseudostarlike of the order* $\alpha = \max\{\alpha_1, \alpha_2\}$.

If the functions $F_j \in D^*$ *are pseudoconvex of the orders* $\alpha_j \in [0,\ (h,\mathbf{b}))$ *then Hadamard composition* $F_1 * F_2$ *is pseudoconvex of the order* $\alpha = \max\{\alpha_1, \alpha_2\}$.

Proof. Since $F_j \in D^*$ that is $f_{(n),j} \le 0$ for all n and j, from (15) it follows that

$$|f_{(n),j}| \le \frac{(h,\mathbf{b}) - \alpha_j}{(\lambda_{(n)},\mathbf{b}) - \alpha_j)} < 1, \quad \|n\| \ge k_0,$$

and, therefore,

$$\sum_{\|n\|=k_0}^{+\infty} \frac{(\lambda_{(n)},\mathbf{b}) - \alpha_1)}{(h,\mathbf{b}) - \alpha_1} |f_{(n),1} f_{(n),2}| \le \sum_{k=1}^{+\infty} \frac{(\lambda_{(n)},\mathbf{b}) - \alpha_1)}{(h,\mathbf{b}) - \alpha_1} |f_{(n),1}| \le 1$$

and similarly

$$\sum_{\|n\|=k_0}^{+\infty} \frac{(\lambda_{(n)},\mathbf{b}) - \alpha_2)}{(h,\mathbf{b}) - \alpha_2} |f_{(n),1} f_{(n),2}| \le 1,$$

i.e. the function $F_1 * F_2$ is pseudostarlike of the order $\alpha = \max\{\alpha_1, \alpha_2\}$.

The proof of the pseudostarlikeness of $F_1 * F_2$ is similar. □

Proposition 2. *Let* $\mathbf{b} > 0$. *If the functions* $F_j \in D^*$ *are pseudostarlike of the orders* $\alpha \in [0,\ (h,\mathbf{b}))$ *and the type* $\beta_j > 0$ *then Hadamard composition* $F_1 * F_2$ *is pseudostarlike of the order* α *and the type* $\beta = \min\{\beta_1,\ \beta_2\}$.

If the functions $F_j \in D^*$ *are pseudoconvex of the orders* $\alpha \in [0,\ (h,\mathbf{b}))$ *and the type* $\beta_j > 0$ *then Hadamard composition* $F_1 * F_2$ *is pseudoconvex of the order* α *and the type* $\beta = \min\{\beta_1,\ \beta_2\}$.

Proof. From (11) it follows that

$$|f_{(n),j}| \le \frac{2\beta_j((h,\mathbf{b}) - \alpha)}{((1+\beta_j)\lambda_{(n)} - (1-\beta_j)h,\mathbf{b}) - 2\beta_j\alpha} < 1, \quad \|n\| \ge k_0$$

and, therefore, as above we have

$$\sum_{\|n\|=k_0}^{+\infty} \frac{((1+\beta_j)\lambda_{(n)} - (1-\beta_j)h,\mathbf{b}) - 2\beta_j\alpha}{2\beta_j((h,\mathbf{b}) - \alpha)} |f_{(n),1}||f_{(n),2}| \le 1, \quad j = 1,\ 2.$$

Hence it follows that $F_1 * F_2$ is pseudostarlike of the order α and the type β_j for each j and, thus, $F_1 * F_2$ is pseudostarlike of the order α and the type $\beta = \min\{\beta_1,\ \beta_2\}$.

The proof of the pseudoconvexity of $F_1 * F_2$ is similar. □

5. Directional differential equation. Note that the entire and analytical solutions of directional differential equations, as well as systems of such equations, were considered, for example, in articles [29–31] (see also in this connection [32–34]).

Here we consider the directional differential equation

$$\partial_{\mathbf{b}}^2 w + (\gamma_0 e^{2(s,h)} + \gamma_1 e^{(s,h)} + \gamma_2) w = 0, \tag{20}$$

where $h = (h_1, \dots, h_p)$, $h_j > 0$ $(1 \le j \le p)$. We will look for a solution to the equation in the form

$$F(s) = e^{(s,h)} + \sum_{\|(\mathbf{n})\|=\|(\mathbf{1})\|}^{+\infty} f_{(\mathbf{n})} \exp\{(\lambda_{(\mathbf{n})}, s)\}, \tag{21}$$

where $\mathbf{n} = (n, ..., n)$, $\lambda_{(\mathbf{1})} = 2h = 2(h_1, ..., h_p)$, $\lambda_{(\mathbf{n})} = \lambda_{(\mathbf{n-1})} + h = (n+1)h$.

Since

$$\partial_{\mathbf{b}}^2 F(s) = (h, \mathbf{b})^2 e^{(s,h)} + \sum_{\|(\mathbf{n})\|=\|(\mathbf{1})\|}^{+\infty} (\lambda_{(\mathbf{n})}, \mathbf{b})^2 f_{(\mathbf{n})} \exp\{(\lambda_{(\mathbf{n})}, s)\},$$

we have

$$(h, \mathbf{b})^2 e^{(s,h)} + \sum_{\|(\mathbf{n})\|=\|(\mathbf{1})\|}^{+\infty} (\lambda_{(\mathbf{n})}, \mathbf{b})^2 f_{(\mathbf{n})} \exp\{(\lambda_{(\mathbf{n})}, s)\} + \gamma_0 e^{3(s,h)} +$$
$$+\gamma_0 \sum_{\|(\mathbf{n})\|=\|(\mathbf{1})\|}^{+\infty} f_{(\mathbf{n})} \exp\{(\lambda_{(\mathbf{n})} + 2h, s)\} + +\gamma_1 e^{2(s,h)} +$$
$$+\gamma_1 \sum_{\|(\mathbf{n})\|=\|(\mathbf{1})\|}^{+\infty} f_{(\mathbf{n})} \exp\{(\lambda_{(\mathbf{n})} + h, s)\} + \gamma_2 e^{(s,h)} + \gamma_2 \sum_{\|(\mathbf{n})\|=\|(\mathbf{1})\|}^{+\infty} f_{(\mathbf{n})} \exp\{(\lambda_{(\mathbf{n})}, s)\} = 0,$$

i.e.

$$(h, \mathbf{b})^2 e^{(s,h)} + \sum_{\|(\mathbf{n})\|=\|(\mathbf{1})\|}^{+\infty} (\lambda_{(\mathbf{n})}, \mathbf{b})^2 f_{(\mathbf{n})} \exp\{(\lambda_{(\mathbf{n})}, s)\} + \gamma_0 e^{3(s,h)} +$$
$$+\gamma_0 \sum_{\|(\mathbf{n})\|=\|(\mathbf{1})\|}^{+\infty} f_{(\mathbf{n})} \exp\{(\lambda_{(\mathbf{n+2})}, s)\} + \gamma_1 e^{2(s,h)} + \gamma_1 \sum_{\|(\mathbf{n})\|=\|(\mathbf{1})\|}^{+\infty} f_{(\mathbf{n})} \exp\{(\lambda_{(\mathbf{n+1})}, s)\} +$$
$$+\gamma_2 e^{(s,h)} + \gamma_2 \sum_{\|(\mathbf{n})\|=\|(\mathbf{1})\|}^{+\infty} f_{(\mathbf{n})} \exp\{(\lambda_{(\mathbf{n})}, s)\} = 0,$$

whence

$$((h, \mathbf{b})^2 + \gamma_2) e^{(s,h)} + \gamma_1 e^{2(s,h)} + \gamma_0 e^{3(s,h)} + \sum_{\|(\mathbf{n})\|=\|(\mathbf{1})\|}^{+\infty} ((\lambda_{(\mathbf{n})} \mathbf{b})^2 + \gamma_2) f_{(\mathbf{n})} \exp\{(\lambda_{(\mathbf{n})}, s)\} +$$
$$+\gamma_1 \sum_{\|(\mathbf{n})\|=\|(\mathbf{2})\|}^{+\infty} f_{((\mathbf{n-1}))} \exp\{(\lambda_{(\mathbf{n})}, s)\} + \gamma_0 \sum_{\|(\mathbf{n})\|=\|(\mathbf{3})\|}^{+\infty} f_{((\mathbf{n-2}))} \exp\{(\lambda_{(\mathbf{n})}, s)\} = 0$$

and, thus,

$$((h, \mathbf{b})^2 + \gamma_2) e^{(s,h)} + \gamma_1 e^{2(s,h)} + \gamma_0 e^{3(s,h)} +$$
$$+((2h, \mathbf{b})^2 + \gamma_2) f_{(\mathbf{1})} \exp\{(2h, s)\} + ((3h, \mathbf{b})^2 + \gamma_2) f_{(\mathbf{2})} \exp\{(3h, s)\} + \gamma_1 f_{((\mathbf{1})} \exp\{(3h, s)\} +$$
$$+ \sum_{\|n\|=\|(\mathbf{3})\|}^{+\infty} \{((\lambda_{(\mathbf{n})}, \mathbf{b})^2 + \gamma_2) f_{(\mathbf{n})} + \gamma_1 f_{((\mathbf{n-1}))} + \gamma_0 f_{((\mathbf{n-2}))}\} \exp\{(\lambda_{(\mathbf{n})}, s)\} = 0.$$

Hence , it follows that

$$(h,\mathbf{b})^2+\gamma_2=0,\quad ((2h,\mathbf{b})^2+\gamma_2)f_{(\mathbf{1})}+\gamma_1=0,\quad ((3h,\mathbf{b})^2+\gamma_2)f_{(\mathbf{2})}+\gamma_1 f_{((\mathbf{1})}+\gamma_0=0$$

and

$$((\lambda_{(\mathbf{n})},\mathbf{b})^2+\gamma_2)f_{(\mathbf{n})}+\gamma_1 f_{((\mathbf{n-1}))}+\gamma_0 f_{((\mathbf{n-2}))}=0,\quad \mathbf{n}\ge\mathbf{3}.$$

Therefore, the following lemma is correct.

Lemma 2. *Function* (2) *satisfies differential equation* (1) *if and only if*

$$\gamma_2=-(h,\mathbf{b})^2,\quad f_{(\mathbf{1})}=-\frac{\gamma_1}{(h,\mathbf{b})^2},\quad f_{(\mathbf{2})}=-\frac{\gamma_1 f_{(\mathbf{1})}+\gamma_0}{2(h,\mathbf{b})^2} \tag{22}$$

and

$$f_{(\mathbf{n})}=-\frac{\gamma_1 f_{(\mathbf{n-1})}+\gamma_0 f_{(\mathbf{n-2})}}{n(h,\mathbf{b})^2},\quad \mathbf{n}\ge\mathbf{3}. \tag{23}$$

Using Theorem 1 and Lemma 2 now we prove the following theorem.

Theorem 5. *Let* $\mathbf{b}>0$, $\gamma_2=-(h,\mathbf{b})^2$, $|\gamma_1|+|\gamma_0|>0$, $0\le\alpha<(h,\mathbf{b})$ *and* $\beta>0$. *Then differential equation* (20) *has an entire solution* (21) *such that:*

1) if $|\gamma_1|+|\gamma_0|<(h,\mathbf{b})^2$ *and*

$$\frac{2|\gamma_1|\{(3\beta+1)(h,\mathbf{b})-2\beta\alpha\}+\{(4\beta+2)(h,\mathbf{b})-2\beta\alpha\}|\gamma_0|}{2(|\gamma_1|+|\gamma_0|)}\le 2\beta((h,\mathbf{b})-\alpha) \tag{24}$$

then function (21) *is pseudostarlike of the order* α *and the type* β *in the direction* $\mathbf{b}\ge 0$.

2) if $0<|\gamma_1|+|\gamma_0|<(h,\mathbf{b})^2/2$ *and*

$$\frac{4\{(3\beta+1)(h,\mathbf{b})-2\beta\alpha\}|\gamma_1|+3\{(4\beta+2)(h,\mathbf{b})-2\beta\alpha\}|\gamma_0|}{4(|\gamma_1|+|\gamma_0|)}\le 2\beta((h,\mathbf{b})-\alpha) \tag{25}$$

then function (21) *is pseudoconvex of the order* α *and the type* β *in the direction* $\mathbf{b}\ge 0$.

Proof. By Theorem 1 function (2) is pseudostarlike of the order α and the type β in the direction $\mathbf{b}\ge 0$ provided

$$\sum_{\|(n)\|=\|(\mathbf{1})\|}^{+\infty}\{((n+2)\beta+n)(h,\mathbf{b})-2\beta\alpha\}|f_{(\mathbf{n})}|\le 2\beta((h,\mathbf{b})-\alpha). \tag{26}$$

On the other hand in view of (3) and (4) we have

$$\sum_{\|(n)\|=\|(\mathbf{1})\|}^{+\infty}\{((n+2)\beta+n)(h,\mathbf{b})-2\beta\alpha\}|f_{(\mathbf{n})}|\le$$

$$\le\{(3\beta+1)(h,\mathbf{b})-2\beta\alpha\}|f_{(\mathbf{1})}|+\{(4\beta+2)(h,\mathbf{b})-2\beta\alpha\}|f_{(\mathbf{2})}|+$$

$$+\sum_{\|(n)\|=\|(\mathbf{3})\|}^{+\infty}\{((n+2)\beta+n)(h,\mathbf{b})-2\beta\alpha\}\frac{|\gamma_1||f_{((\mathbf{n-1}))}|+|\gamma_0||f_{((\mathbf{n-2}))}|}{n(h,\mathbf{b})^2}=$$

$$=\{(3\beta+1)(h,\mathbf{b})-2\beta\alpha\}|f_{(\mathbf{1})}|+\{(4\beta+2)(h,\mathbf{b})-2\beta\alpha\}|f_{(\mathbf{2})}|+$$

$$+\sum_{\|(n)\|=\|(\mathbf{2})\|}^{+\infty}\{((n+3)\beta+n+1)(h,\mathbf{b})-2\beta\alpha\}\frac{|\gamma_1||f_{((\mathbf{n}))}|}{(n+1)(h,\mathbf{b})^2}=$$

$$+\sum_{\|(n)\|=\|(\mathbf{1})\|}^{+\infty}\{((n+4)\beta+n+2)(h,\mathbf{b})-2\beta\alpha\}\frac{|\gamma_0||f_{((\mathbf{n}))}|}{(n+2)(h,\mathbf{b})^2}=$$

$$=\{(3\beta+1)(h,\mathbf{b})-2\beta\alpha\}|f_{(\mathbf{1})}|+\{(4\beta+2)(h,\mathbf{b})-2\beta\alpha\}|f_{(\mathbf{2})}|-$$

$$(4\beta+2)(h,\mathbf{b})-2\beta\alpha\}\frac{|\gamma_1||f_{((\mathbf{1}))}|}{2(h,\mathbf{b})^2}+$$

$$+\sum_{\|(n)\|=\|(\mathbf{1})\|}^{+\infty}\{((n+3)\beta+n+1)(h,\mathbf{b})-2\beta\alpha\}\frac{|\gamma_1||f_{((\mathbf{n}))}|}{(n+1)(h,\mathbf{b})^2}=$$

$$=\sum_{\|(n)\|=\|(\mathbf{1})\|}^{+\infty}\{((n+4)\beta+n+2)(h,\mathbf{b})-2\beta\alpha\}\frac{|\gamma_0||f_{((\mathbf{n}))}|}{(n+2)(h,\mathbf{b})^2}=$$

$$=\{(3\beta+1)(h,\mathbf{b})-2\beta\alpha\}|f_{(\mathbf{1})}|+\{(4\beta+2)(h,\mathbf{b})-2\beta\alpha\}\frac{|\gamma_0|}{2(h,\mathbf{b})^2}+$$

$$+\sum_{\|(n)\|=\|(\mathbf{1})\|}^{+\infty}\Big(\{((n+3)\beta+n+1)(h,\mathbf{b})-2\beta\alpha\}\frac{|\gamma_1|}{(n+1)(h,\mathbf{b})^2}+$$

$$+\{((n+4)\beta+n+2)(h,\mathbf{b})-2\beta\alpha\}\frac{|\gamma_0|}{(n+2)(h,\mathbf{b})^2}\Big)|f_{((\mathbf{n}))}|\le$$

$$\le\frac{2|\gamma_1|\{(3\beta+1)(h,\mathbf{b})-2\beta\alpha\}+\{(4\beta+2)(h,\mathbf{b})-2\beta\alpha\}|\gamma_0|}{2(h,\mathbf{b})^2}+$$

$$+\sum_{\|(n)\|=\|(\mathbf{1})\|}^{+\infty}\Big(2\frac{|\gamma_1|\{((n+2)\beta+n)(h,\mathbf{b})-2\beta\alpha\}}{(n+1)(h,\mathbf{b})^2}+$$

$$+3\frac{|\gamma_0|\{((n+2)\beta+n)(h,\mathbf{b})-2\beta\alpha\}}{(n+2)(h,\mathbf{b})^2}\Big)|f_{((\mathbf{n}))}|\le$$

$$\le\frac{2|\gamma_1|\{(3\beta+1)(h,\mathbf{b})-2\beta\alpha\}+\{(4\beta+2)(h,\mathbf{b})-2\beta\alpha\}|\gamma_0|}{2(h,\mathbf{b})^2}+$$

$$+\frac{|\gamma_1|+|\gamma_0|}{(h,\mathbf{b})^2}\sum_{\|(n)\|=\|(\mathbf{1})\|}^{+\infty}\{((n+2)\beta+n)(h,\mathbf{b})-2\beta\alpha\}|f_{((\mathbf{n}))}|.$$

Therefore, if $|\gamma_1|+|\gamma_0|<(h,\mathbf{b})^2$ then

$$\Big(1-\frac{|\gamma_1|+|\gamma_0|}{(h,\mathbf{b})^2}\Big)\sum_{\|(n)\|=\|(\mathbf{1})\|}^{+\infty}\{((n+2)\beta+n)(h,\mathbf{b})-2\beta\alpha\}|f_{((\mathbf{n})}|\le$$

$$\le\frac{2|\gamma_1|\{(3\beta+1)(h,\mathbf{b})-2\beta\alpha\}+\{(4\beta+2)(h,\mathbf{b})-2\beta\alpha\}|\gamma_0|}{2(h,\mathbf{b})^2}$$

and in view of (24) we get (26), i.e. function (21) is pseudostarlike of the order α and the type β in the direction $\mathbf{b}\ge 0$.

Then by Proposition 2 function (21) is pseudoconvex of the order α and the type β in the direction $\mathbf{b}\ge 0$ provided

$$\sum_{\|(n)\|=\|(\mathbf{1})\|}^{+\infty}(n+1)\{((n+2)\beta+n)(h,\mathbf{b})-2\beta\alpha\}|f_{(\mathbf{n})}|\le 2\beta((h,\mathbf{b})-\alpha) \qquad (27)$$

On the other hand in view of (22) and (23), as above, we have

$$\sum_{\|(n)\|=\|(\mathbf{1})\|}^{+\infty}(n+1)\{((n+2)\beta+n)(h,\mathbf{b})-2\beta\alpha\}|f_{(\mathbf{n})}|\le$$

$$\le 2\{(3\beta+1)(h,\mathbf{b})-2\beta\alpha\}|f_{(\mathbf{1})}|+3\{(4\beta+2)(h,\mathbf{b})-2\beta\alpha\}|f_{(\mathbf{2})}|+$$

$$+\sum_{\|(n)\|=\|(\mathbf{2})\|}^{+\infty}(n+2)\{((n+3)\beta+n+1)(h,\mathbf{b})-2\beta\alpha\}\frac{|\gamma_1||f_{((\mathbf{n})}|}{(n+1)(h,\mathbf{b})^2}=$$

$$=\sum_{\|(n)\|=\|(\mathbf{1})\|}^{+\infty}(n+3)\{((n+4)\beta+n+2)(h,\mathbf{b})-2\beta\alpha\}\frac{|\gamma_0||f_{((\mathbf{n})}|}{(n+2)(h,\mathbf{b})^2}\le$$

$$\le\frac{4\{(3\beta+1)(h,\mathbf{b})-2\beta\alpha\}|\gamma_1|+3\{(4\beta+2)(h,\mathbf{b})-2\beta\alpha\}|\gamma_0|}{2(h,\mathbf{b})^2}+$$

$$+\sum_{\|(n)\|=\|(\mathbf{1})\|}^{+\infty}\Big(4\frac{|\gamma_1|(n+1)\{((n+2)\beta+n)(h,\mathbf{b})-2\beta\alpha\}}{(n+1)(h,\mathbf{b})^2}+$$

$$+6\frac{|\gamma_0|(n+1)\{((n+2)\beta+n)(h,\mathbf{b})-2\beta\alpha\}}{(n+2)(h,\mathbf{b})^2}\Big)|f_{((\mathbf{n})}|\le$$

$$\le\frac{4\{(3\beta+1)(h,\mathbf{b})-2\beta\alpha\}|\gamma_1|+3\{(4\beta+2)(h,\mathbf{b})-2\beta\alpha\}|\gamma_0|}{2(h,\mathbf{b})^2}+$$

$$+2\frac{|\gamma_1|+|\gamma_0|}{(h,\mathbf{b})^2}\sum_{\|(n)\|=\|(\mathbf{1})\|}^{+\infty}(n+1)\{((n+2)\beta+n)(h,\mathbf{b})-2\beta\alpha\}|f_{((\mathbf{n})}|$$

and in view of (25) we get (27), i.e. function (21) is pseudoconvex of the order α and the type β in the direction $\mathbf{b}\ge 0$.

Finally, for every $\sigma\in\mathbb{R}^p$ there exists $n^0=n^0(\sigma)$ such that for all $n\ge n^0$ one has

$$\frac{(|\gamma_0|+|\gamma_1|)\exp\{2(h,\sigma)\}}{n(h,\mathbf{b})^2}\le\frac{1}{2}.$$

Therefore, we have, as above,

$$\sum_{\|(\mathbf{n})\|=\|(\mathbf{n^0})\|}^{+\infty} |f_{(\mathbf{n})}| \exp\{(\lambda_{(\mathbf{n})}, \sigma)\} \leq$$

$$\leq \sum_{\|(\mathbf{n})\|=\|(\mathbf{n^0})\|}^{+\infty} \frac{|\gamma_1|}{n(h,\mathbf{b})^2} |f_{(\mathbf{n-1})}| \exp\{(\lambda_{(\mathbf{n-1})}, \sigma)\} \exp\{(\lambda_{(\mathbf{n})}, \sigma) - (\lambda_{(\mathbf{n-1})}, \sigma)\} +$$

$$+ \sum_{\|(\mathbf{n})\|=\|(\mathbf{n^0})\|}^{+\infty} \frac{|\gamma_0|}{n(h,\mathbf{b})^2} |f_{(\mathbf{n-2})}| \exp\{(\lambda_{(\mathbf{n-2})}, \sigma)\} \exp\{(\lambda_{(\mathbf{n})}, \sigma) - (\lambda_{(\mathbf{n-2})}, \sigma)\} =$$

$$= \sum_{\|(\mathbf{n})\|=\|(\mathbf{n^0-1})\|}^{+\infty} \frac{|\gamma_1|}{(n+1)(h,\mathbf{b})^2} |f_{(\mathbf{n})}| \exp\{(\lambda_{(\mathbf{n})}, \sigma)\} \exp\{(h, \sigma)\} +$$

$$+ \sum_{\|(\mathbf{n})\|=\|(\mathbf{n^0-2})\|}^{+\infty} \frac{|\gamma_0|}{(n+2)(h,\mathbf{b})^2} |f_{(\mathbf{n})}| \exp\{(\lambda_{(\mathbf{n})}, \sigma)\} \exp\{2(h, \sigma)\} \leq$$

$$\leq \frac{|\gamma_1|}{n(h,\mathbf{b})^2} |f_{(\mathbf{n^0-1})}| \exp\{(\lambda_{(\mathbf{n^0-1})}, \sigma)\} \exp\{(h, \sigma)\} +$$

$$+ \sum_{\|(\mathbf{n})\|=\|(\mathbf{n^0})\|}^{+\infty} \frac{|\gamma_1| + |\gamma_0|}{n(h,\mathbf{b})^2} |f_{(\mathbf{n})}| \exp\{(\lambda_{(\mathbf{n})}, \sigma)\} \exp\{2(h, \sigma)\} +$$

$$+ \frac{|\gamma_0|}{n(h,\mathbf{b})^2} |f_{(\mathbf{n^0-2})}| \exp\{(\lambda_{(\mathbf{n^0-2})}, \sigma)\} \exp\{2(h, \sigma)\} \leq \frac{1}{2} \sum_{\|(\mathbf{n})\|=\|(\mathbf{n^0})\|}^{+\infty} |f_{(\mathbf{n})}| \exp\{(\lambda_{(\mathbf{n})}, \sigma)\} + \text{const},$$

i.e. Dirichlet series (21) is entire (absolutely convergent in $\mathbb{C}^p$). The proof of Theorem 5 is complete d. □

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