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EXISTENCE OF BASIC SOLUTIONS OF FIRST ORDER LINEAR HOMOGENEOUS SET-VALUED DIFFERENTIAL EQUATIONS

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The paper presents various derivatives of set-valued mappings, their main properties and how they are related to each other. Next, we consider Cauchy problems with linear homogeneous set-valued differential equations with different types of derivatives (Hukuhara derivative, PS-derivative and BG-derivative). It is known that such initial value problems with PS-derivative and BG-derivative have infinitely many solutions. Two of these solutions are called basic. These are solutions such that the diameter function of the solution section is a monotonically increasing (the first basic solution) or monotonically decreasing (the second basic solution) function. However, the second basic solution does not always exist. We provide conditions for the existence of basic solutions of such initial value problems. It is shown that their existence depends on the type of derivative, the matrix of coefficients on the right-hand and the type of the initial set. Model examples are considered.

1. Introduction. The set-valued differential, integral and discrete-time equations and inclusions are an important part of the theory of set-valued analysis, and they are high-valued for the control theory and its applications, as well as for fuzzy differential equations. They were first introduced in 1969 by F. S. de Blasi and F. Iervolino [7]. Later, set-valued differential equations have been studied by many scientists due to their applications in many areas. A lot of results on the theory of set-valued differential, integral and discrete-time equations and inclusions can be found in the following books and papers [11, 16, 17, 20, 21, 24, 29, 31, 36, 41] and references therein.

In this paper first we consider some definitions of the derivative of a set-valued mapping (Hukuhara derivative [13], Plotnikov-Skripnik derivative [32] and Bede-Gal derivative [2, 25, 26, 43, 44]) and some of their properties. Next, we consider a linear homogeneous set-valued differential equation with different types of derivatives (Hukuhara derivative, PS-derivative and BG-derivative). In [18,19] it is proved that the Cauchy problem for differential equations with PS-derivative and BG-derivative has infinitely many solutions. Two of these solutions are called basic: solutions such that the diameter function of the solution section is a monotonically increasing (the first basic solution) or monotonically decreasing (the second basic solution) function. However, the second base solution does not always exist. Here we will justify the conditions for the existence of basic solutions of such initial value problems and show that their existence depends on the type of derivative, the matrix of coefficients on the right side and the type of the initial set.

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2. Preliminaries. In this section we recall some results from the literature that are of interest for our paper.

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}^n$ be the n-dimensional Euclidean space ($n \geq 2$). Denote by $\text{conv}(\mathbb{R}^n)$ the *set of nonempty compact and convex subsets* of $\mathbb{R}^n$ with the *Hausdorff metric*

$$h(X,Y) = \max\left\{ \sup_{x\in X}\inf_{y\in Y}\|x-y\|, \quad \sup_{y\in Y}\inf_{x\in X}\|x-y\| \right\},$$

where $X, Y \in \text{conv}(\mathbb{R}^n)$, $\|\cdot\|$ denotes the Euclidean norm.

In addition to the usual set-theoretic operations, we introduce two operations in the space $\text{conv}(\mathbb{R}^n)$: the sum of the sets, the product of the scalar on the set and the operation of the product of the matrix on the set:

$$X+Y = \{x+y \,:\, x\in X, \vee y \in Y\}, \quad \lambda X = \{\lambda x \,:\, x\in X\}, \quad AX = \{Ax \,:\, x\in X\},$$

where $\lambda \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$.

The following theorem will be used further:

Theorem 1 ([10, 12]). *For any real $(n\times n)$-matrix A there exist two orthogonal $(n\times n)$-matrices U and V such that $U^TAV = \Sigma$, where Σ is a diagonal matrix. We can also choose matrices U and V such that the diagonal elements of the matrix Σ satisfy the condition $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r > \sigma_{r+1} = ... = \sigma_n = 0$, where r is the rank of the matrix A. That is, if A is a nondegenerate matrix, then $\sigma_1 \geq ... \geq \sigma_n > 0$.*

Therefore, this matrix A can be represented as $A = U\Sigma V^T$. This decomposition is called a *singular decomposition.* Columns $\mathbf{u_1}, ..., \mathbf{u_n}$ of matrix U are called the *left singular vectors*, columns $\mathbf{v_1}, ..., \mathbf{v_n}$ of matrix V are called the *right singular vectors*, and the numbers $\sigma_1, ..., \sigma_n$ are called the *singular numbers* (s-numbers) of the matrix A.

One can easily prove the following proposition.

Proposition 1. *If a matrix $A \in \mathbb{R}^{2\times 2}$ is such that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then*

$$\sigma_1 = \sqrt{(a^2+b^2+c^2+d^2+\sqrt{F})/2}, \quad \sigma_2 = \sqrt{(a^2+b^2+c^2+d^2-\sqrt{F})/2},$$

where $F = (a^2+b^2-c^2-d^2)^2 + 4(ac+bd)^2$. Consequently, if $F = 0$, then the singular numbers σ_1 and σ_2 of the matrix A coincide and are equal to $\sqrt{(a^2+b^2+c^2+d^2)/2}$.

Let $A \in \mathbb{R}^{n\times n}$ and $B_r(\mathbf{c}) = \{x \in \mathbb{R}^n \,:\, \|x-c\| \leq r\}$ is the closed ball of radius $r > 0$ centered at the point $\mathbf{c} \in \mathbb{R}^n$. By [10], if $\text{rank}(A) = k$, then the set $Y = AB_1(\mathbf{0})$ is a k-dimensional ellipsoid, its axis lengths are equal to the corresponding singular numbers of the matrix A, where $\mathbf{0} = (0, ..., 0)^T$ is the zero vector. Also, if $\text{rank}(A) = n$, then

$$B_{\sigma_n}(\mathbf{0}) \subset Y \subset B_{\sigma_1}(\mathbf{0}),$$

where $B_{\sigma_n}(\mathbf{0})$ is the inscribed ball in the set Y (i.e. the largest ball $B_r(\mathbf{0})$ that can fit inside the set Y), $B_{\sigma_1}(\mathbf{0})$ is the smallest circumscribed ball of the set Y (i.e. the smallest ball $B_r(\mathbf{0})$, such that $Y \subseteq B_r(\mathbf{0})$).

It is also easy to see that if A is an orthogonal matrix, then $AB_r(\mathbf{0}) \equiv B_r(\mathbf{0})$ for all $r > 0$.

Lemma 1 ([38]). *The following properties hold: 1) $(\text{conv}(\mathbb{R}^n), h)$ is a complete metric space; 2) $h(X+Z, Y+Z) = h(X,Y)$; 3) $h(\lambda X, \lambda Y) = |\lambda| h(X,Y)$ for all $X, Y, Z \in \text{conv}(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$.*

However, conv $(\mathbb{R}^n)$ is not a linear space because it does not contain inverse elements for the addition, and therefore the difference is not well defined, i.e. if $X \in$ conv $(\mathbb{R}^n)$ and $X \neq \{\mathbf{x}\}$, then $X + (-1)X \neq \{\mathbf{0}\}$. As a consequence, alternative formulations for difference have been suggested [3,13,30,38,40]. One of these alternatives is the Hukuhara difference [13].

Let $X, Y \in$ conv $(\mathbb{R}^n)$. A set $Z \in$ conv $(\mathbb{R}^n)$ such that $X = Y + Z$ is called a *Hukuhara difference* (H-difference) ([13]) of the sets X and Y and is denoted by $X \overset{H}{-} Y$.

Obviously, $X \overset{H}{-} Y \neq X + (-1)Y$. In this case $X \overset{H}{-} X = \{\mathbf{0}\}$ and $(X + Y) \overset{H}{-} Y = X$ for any $X, Y \in$ conv $(\mathbb{R}^n)$. Properties of this difference are studied in details in [13, 17, 20, 29, 31, 36, 38, 40]. M. Hukuhara introduced the concept of H-differentiability for set-valued functions by using the H-difference [13].

Let $X\colon [0, T] \to$ conv $(\mathbb{R}^n)$ be a set-valued mapping; $(t_0 - \Delta, t_0 + \Delta) \subset [0, T]$ be a Δ-neighborhood of a point $t_0 \in [0, T]; \Delta > 0$. For any $\rho \in (0, \Delta)$ consider the following Hukuhara differences $X(t_0 + \rho) \overset{H}{-} X(t_0)$ and $X(t_0) \overset{H}{-} X(t_0 - \rho)$ if these differences exist.

We say that the mapping $X\colon [0, T] \to$ conv $(\mathbb{R}^n)$ has *Hukuhara derivative* (H-derivative) ([13]) $D_H X(t_0)$ at a point $t_0 \in (0, T)$, if there exists an element $D_H X(t_0) \in$ conv $(\mathbb{R}^n)$ such that the limits

$$\lim_{\rho \to 0^+} \rho^{-1}(X(t_0 + \rho) \overset{H}{-} X(t_0)) \quad \text{and} \quad \lim_{\rho \to 0^+} \rho^{-1}(X(t_0) \overset{H}{-} X(t_0 - \rho))$$

exist in the topology of conv $(\mathbb{R}^n)$ and are equal to $D_H X(t_0)$.

If H-derivative $D_H X(t)$ exists for all $t \in (0, T)$ and the limits

$$\lim_{\rho \to 0^+} \rho^{-1}(X(\rho) \overset{H}{-} X(0)) \quad \text{and} \quad \lim_{\rho \to 0^+} \rho^{-1}(X(T) \overset{H}{-} X(T - \rho))$$

exist, then we say that the *set-valued mapping* $X(\cdot)$ *is* H-*differentiable* on $[0, T]$.

The properties of the Hukuhara derivative are obtained in [8, 17, 29, 31, 36, 38]. Here, we mention some of them.

Theorem 2 ([13]). *If the mapping* $X\colon [0, T] \to$ conv $(\mathbb{R}^n)$ *is H-differentiable on* $[0, T]$, *then* $X(t) = X(0) + \int_0^t D_H X(s) ds$, *where the integral is understood in the sense of [13].*

Corollary 1. *If the mapping* $X(\cdot)$ *is H-differentiable on* $[0, T]$, *then function* $\mathrm{diam}(X(\cdot))$ *is a non-decreasing function on* $[0, T]$, *where* $\mathrm{diam}(X(t)) = \max\limits_{x, y \in X(t)} \|x - y\|$.

Corollary 2. *If the function* $\mathrm{diam}(X(\cdot))$ *is a decreasing function on* $[0, T]$, *then the mapping* $X(\cdot)$ *is not H-differentiable on* $[0, T]$.

Later, T.F. Bridgland introduced the concept of a derivative for set-valued mappings without using the Hukuhara difference and considered its properties [8].

We say that the mapping $X\colon [0, T] \to$ conv $(\mathbb{R}^n)$ has *Huygens derivative* $D_B X(t_0)$ ([8]) at a point $t_0 \in (0, T)$, if there exists an element $D_B X(t_0) \in$ conv $(\mathbb{R}^n)$ such that

$$\lim_{\rho \to 0^+} \rho^{-1} h(X(t_0 + \rho), X(t_0) + \rho D_B X(t_0)) = 0,$$
$$\lim_{\rho \to 0^+} \rho^{-1} h(X(t_0), X(t_0 - \rho) + \rho D_B X(t_0)) = 0.$$

A similar derivative was later considered in [6, 14, 15, 22, 27, 28, 36]. The examples can be constructed where the Huygens derivative exists, but the Hukuhara derivative does not exist (see [6, 36]). However, if the diameter of the set-valued mapping is a decreasing function, the Huygens derivative also does not exist [8, 36].

The last drawback of these derivatives significantly impairs the possibility of using these derivative for modeling applied processes. Later, to overcome the shortcomings of this approach, other types of differences and derivatives for set-valued mappings were considered:

π-derivative [3, 42] and T-derivative [30, 31, 36]. However, difficulties arose in writing the corresponding set-valued differential equation with these derivatives.

Later, A. V. Plotnikov and N. V. Skripnik took advantage of some approaches that were used in [30] and introduced a new definition of a derivative, which eliminated the main drawback of the Hukuhara derivative.

Let $X\colon [0,T] \to \operatorname{conv}(\mathbb{R}^n)$ and $t \in (0,T)$. We say that $X(\cdot)$ has a PS-*derivative* $D_{ps}X(t) \in \operatorname{conv}(\mathbb{R}^n)$ at $t \in (0,T)$ ([32]), if for all $\rho > 0$ that are sufficiently close to 0, the H-differences and the limits exist in at least one of the following expressions:

(i) $\lim\limits_{\rho\to 0} \rho^{-1}(X(t+\rho) \overset{H}{-} X(t)) = \lim\limits_{\rho\to 0} \rho^{-1}(X(t) \overset{H}{-} X(t-\rho)) = D_{ps}X(t)$,

(ii) $\lim\limits_{\rho\to 0} \rho^{-1}(X(t) \overset{H}{-} X(t+\rho)) = \lim\limits_{\rho\to 0} \rho^{-1}(X(t-\rho) \overset{H}{-} X(t)) = D_{ps}X(t)$,

(iii) $\lim\limits_{\rho\to 0} \rho^{-1}(X(t+\rho) \overset{H}{-} X(t)) = \lim\limits_{\rho\to 0} \rho^{-1}(X(t-\rho) \overset{H}{-} X(t)) = D_{ps}X(t)$,

(iv) $\lim\limits_{\rho\to 0} \rho^{-1}(X(t) \overset{H}{-} X(t+\rho)) = \lim\limits_{\rho\to 0} \rho^{-1}(X(t) \overset{H}{-} X(t-\rho)) = D_{ps}X(t)$.

Properties of the PS-derivative are obtained in [18, 19, 32–35]. Here, we mention some of them.

Remark 1 ([32])**.** If the set-valued mapping $X(\cdot)$ is H-differentiable then it is PS-differentiable and $D_{ps}X(t) = D_H X(t)$.

Remark 2 ([32])**.** If the set-valued mapping $X(\cdot)$ is PS-differentiable on $[0,T]$ and $\operatorname{diam}(X(\cdot))$ is a non-decreasing function on $[0,T]$ then the set-valued mapping $X(\cdot)$ is H-differentiable and $D_{ps}X(t) = D_H X(t)$.

Remark 3 ([32])**.** There exist set-valued mappings that are PS-differentiable but not H-differentiable.

Theorem 3 ([32])**.** *If the mapping* $X\colon [0,T] \to \operatorname{conv}(\mathbb{R}^n)$ *is PS-differentiable on* $[0,T]$*, then for all* $t \in [0,T]$*:*

(i) if the function $\operatorname{diam}(X(t))$ *is a non-decreasing function on* $[0,T]$*, then:*
$$X(t) = X(0) + \int_0^t D_{ps}X(s)ds;$$
(ii) if the function $\operatorname{diam}(X(t))$ *is a decreasing function on* $[0,T]$*, then*
$$X(t) = X(0) \overset{H}{-} \int_0^t D_{ps}X(s)ds.$$

Simultaneously, M. T. Malinowski [25, 26], H. Vu and L. S. Dong [43], H. Vu and N. Van Hoa [44] and Ş. E. Amrahov, A. Khastan, N. Gasilov and A .G. Fatullayev [2] adapted the concept of the Bede-Gal derivative [4, 39] for interval-valued mappings on set-valued mappings.

Definition 1 ([2, 43])**.** Let $X\colon [0,T] \to \operatorname{conv}(\mathbb{R}^n)$ and $t \in (0,T)$. We say that $X(\cdot)$ has a BG-*derivative* $D_{bg}X(t) \in \operatorname{conv}(\mathbb{R}^n)$ at $t \in (0,T)$, if for all $\rho > 0$ that are sufficiently close to 0, the H-differences and the limits exist in at least one of the following expressions:

(i) $\lim\limits_{\rho\to 0} \rho^{-1}(X(t+\rho) \overset{H}{-} X(t)) = \lim\limits_{\rho\to 0} \rho^{-1}(X(t) \overset{H}{-} X(t-\rho)) = D_{bg}X(t)$,

(ii) $\lim\limits_{\rho\to 0} (-\rho)^{-1}(X(t) \overset{H}{-} X(t+\rho)) = \lim\limits_{\rho\to 0} (-\rho)^{-1}(X(t-\rho) \overset{H}{-} X(t)) = D_{bg}X(t)$,

(iii) $\lim\limits_{\rho\to 0} \rho^{-1}(X(t+\rho) \overset{H}{-} X(t)) = \lim\limits_{\rho\to 0} (-\rho)^{-1}(X(t-\rho) \overset{H}{-} X(t)) = D_{bg}X(t)$,

(iv) $\lim\limits_{\rho\to 0} (-\rho)^{-1}(X(t) \overset{H}{-} X(t+\rho)) = \lim\limits_{\rho\to 0} \rho^{-1}(X(t) \overset{H}{-} X(t-\rho)) = D_{bg}X(t)$.

Remark 4. In [25,26] M. T. Malinowski considered set-valued mappings that satisfy condition (ii) and called this derivative the second type Hukuhara derivative.

Remark 5 ([2,43])**.** If the set-valued mapping $X(\cdot)$ is H-differentiable on $[0,T]$ it is BG-differentiable on $[0,T]$ and $D_{bg}X(t) = D_H X(t)$.

Remark 6 ([2,43])**.** If the set-valued mapping $X(\cdot)$ is BG-differentiable on $[0,T]$ and the function $\operatorname{diam}(X(\cdot))$ is a non-decreasing function on $[0,T]$ then the set-valued mapping $X(\cdot)$ is H-differentiable and $D_{bg}X(t) = D_H X(t)$.

Remark 7 ([2,43])**.** There exist set-valued mappings that are BG-differentiable but not H-differentiable.

Theorem 4 ([2])**.** *If the mapping* $X\colon [0,T] \to \operatorname{conv}(\mathbb{R}^n)$ *is BG-differentiable on* $[0,T]$, *then for all* $t \in [0,T]$:

(i) if the function $\operatorname{diam}(X(t))$ *is a non-decreasing function on* $[0,T]$, *then*
$$X(t) = X(0) + \int_0^t D_{bg}X(s)ds;$$
(ii) if the function $\operatorname{diam}(X(t))$ *is a decreasing function on* $[0,T]$, *then*
$$X(t) = X(0) \overset{H}{-} (-1)\int_0^t D_{bg}X(s)ds.$$

Remark 8. By Remarks 1 and 5, if the set-valued mapping $X(\cdot)$ is H-differentiable on $[0,T]$ then it is BG-differentiable on $[0,T]$ and PS-differentiable on $[0,T]$ as well as $D_H X(t) = D_{ps}X(t) = D_{bg}X(t)$.

Remark 9 ([18,19])**.** There exist set-valued mappings $X(\cdot)$ such that $D_{bg}X(t) \neq D_{ps}X(t)$ for any t.

3. Initial value problem with linear set-valued differential equations. In this section, we consider the Cauchy problem
$$DX(t) = AX(t), \quad X(0) = X_0, \tag{1}$$
where $A \in \mathbb{R}^{n\times n}$ is a nondegenerate matrix, $X\colon [0,T] \to \operatorname{conv}(\mathbb{R}^n)$ is a set-valued mapping, $DX(t)$ is one of the previously considered derivatives $(D_H X(t), D_{ps}X(t), D_{bg}(t))$ of the set-valued mapping $X(t)$, $n \geq 2$.

Definition 2. A set-valued mapping $X(\cdot)$ is called a *solution of problem* (1) if it is continuously differentiable and satisfies system (1) everywhere on $[0,T]$.

Remark 10. In paper [18] a differential equation $DX = aX,\ X(0) = X_0$ was considered, where a is a real number.

As known, the Cauchy problem with differential equation with Hukuhara derivative
$$D_H X(t) = AX(t), \quad X(0) = X_0 \tag{2}$$
has a unique solution on the interval $[0,T]$ ([29,31]). It is also obvious that the function $\operatorname{diam}(X(t))$ is a non-decreasing function on $[0,T]$.

Remark 11. The solution of problem (2) is a solution of the integral equation

$$X(t) = X_0 + \int_0^t AX(s)ds, \ t \in [0, T]$$

and vice versa.

Remark 12 ([7, 29, 31])**.** If $A = aE$ then $X(t) = e^{at}X_0$ for all $t \in [0, T]$, where $a \geq 0$ is a number, E is the identity matrix.

Now, we consider the Cauchy problem (1) with linear differential equation with PS-derivative and BG-derivative. By [2, 32–35], this initial value problem has at least one solution. Moreover, one of these solutions (the one whose diameter is a non-decreasing function) coincides with the solution of the corresponding problem (2).

We will show it by the following example.

Example 1. Let

$$DX(t) = AX(t), \ X(0) = B_1(\mathbf{0}), \ t \in [0, 1], \tag{3}$$

where $A \in \mathbb{R}^{2\times 2}$ such that $A = \begin{pmatrix} a\cos(\phi) & -a\sin(\phi) \\ a\sin(\phi) & a\cos(\phi) \end{pmatrix}$, $a \in \mathbb{R}$ and $\phi \in [0, 2\pi)$ are numbers and $a \neq 0$, $X\colon [0,1] \to \text{conv}\,(\mathbb{R}^2)$ is a set-valued mapping, $DX(t)$ is one of the previously considered derivatives $(D_H X(t), D_{ps}X(t), D_{bg}(t))$ of the set-valued mapping $X(t)$.

Obviously, the matrix A is such that $A = a\,R(\phi)$, where a is the number, $R(\phi)$ is the rotation matrix. In this case, the singular values of the matrix A are equal to each other for any a and ϕ and $\sigma_1 = \sigma_2 = |a|$. By [10, 38] we have

$$AB_1(\mathbf{0}) = aR(\phi)B_1(\mathbf{0}) = |a|B_1(\mathbf{0}) = B_{|a|}(\mathbf{0}).$$

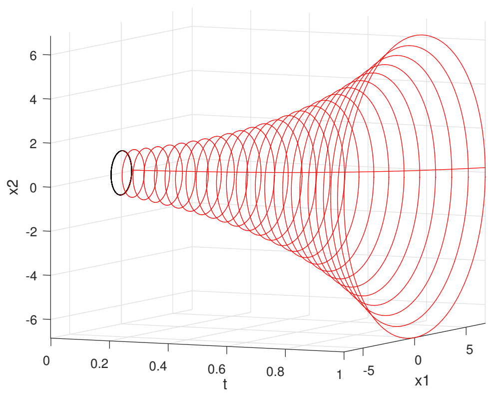

Figure 1: $a = 2$, $X_1(t)$, $t \in [0, 1]$

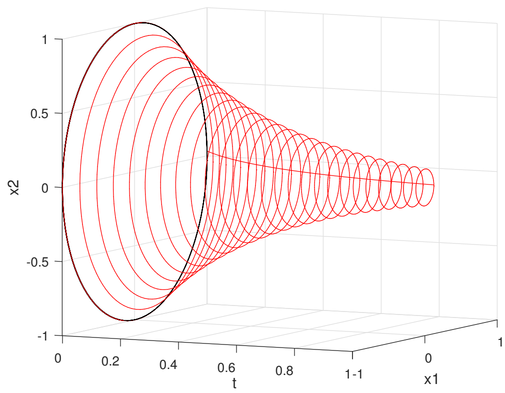

Figure 2: $a = 2$, $X_2(t)$, $t \in [0, 1]$

Hence, the set-valued mapping $X(t) = B_{e^{|a|t}}(\mathbf{0})$ is a solution of the Cauchy problem (3) with Hukuhara differential equation (see Figure 1). This can be verified by substituting $X(t) = B_{e^{|a|t}}(\mathbf{0})$ into system (3), i.e

$$D_H X(t) = AX(t) \implies D_H B_{e^{|a|t}}(\mathbf{0}) = AB_{e^{|a|t}}(\mathbf{0}) \implies$$
$$D_H(e^{|a|t}B_1(\mathbf{0})) = aR(\phi)B_{e^{|a|t}}(\mathbf{0}) \implies \frac{d(e^{|a|t})}{dt}B_1(\mathbf{0}) = |a|B_{e^{|a|t}}(\mathbf{0}) \implies$$
$$|a|e^{|a|t}B_1(\mathbf{0}) \equiv |a|e^{|a|t}B_1(\mathbf{0}) \quad \text{and} \quad X(0) = B_{e^{|a|0}}(\mathbf{0}) \equiv B_1(\mathbf{0}).$$

Also the set-valued mapping $X_1(t) = X(t)$ is a solution of the Cauchy problem (3) for the differential equation with the PS-derivative and the Cauchy problem (3) for the differential equation with the BG-derivative and is called the first basic solution (see Figure 1). Consequently, the function $\mathrm{diam}(X_1(t))$ is an increasing function on $[0,1]$.

The set-valued mapping $X_2(t) = B_{e^{-|a|t}}(\mathbf{0})$ is a solution of the Cauchy problem (3) for the differential equation with the PS-derivative and the Cauchy problem (3) for the differential equation with the BG-derivative and is called the second basic solution (see Figure 2). Consequently, the function $\mathrm{diam}(X_2(t))$ is a decreasing function on $[0,1]$. By [2, 32], the second basic solution $X_2(\cdot)$ is also a solution of the corresponding integral equation:

$$X_2^{ps}(t) = X_0 \overset{H}{-} \int_0^t AX_2^{ps}(s)ds \quad \text{or} \quad X_2^{bg}(t) = X_0 \overset{H}{-} (-1)\int_0^t AX_2^{bg}(s)ds.$$

Let us prove that $X_2(t) = B_{e^{-|a|t}}(\mathbf{0})$ is a solution of the Cauchy problem (3) for the differential equation with the PS-derivative and differential equation with the BG-derivative:

PS-derivative	**BG-derivative**
$X_0 \overset{H}{-} \int\limits_0^t AX_2^{ps}(s)ds =$	$X_0 \overset{H}{-} (-1) \int\limits_0^t AX_2^{bg}(s)ds =$
$B_1(\mathbf{0}) \overset{H}{-} \int\limits_0^t AB_{e^{-\|a\|s}}(\mathbf{0})ds =$	$B_1(\mathbf{0}) \overset{H}{-} (-1) \int\limits_0^t AB_{e^{-\|a\|s}}(\mathbf{0})ds =$
$B_1(\mathbf{0}) \overset{H}{-} \int\limits_0^t aR(\phi)B_{e^{-\|a\|s}}(\mathbf{0})ds =$	$B_1(\mathbf{0}) \overset{H}{-} (-1) \int\limits_0^t aR(\phi)B_{e^{-\|a\|s}}(\mathbf{0})ds =$
$B_1(\mathbf{0}) \overset{H}{-} \int\limits_0^t \|a\|e^{-\|a\|s}dsB_1(\mathbf{0}) =$	$B_1(\mathbf{0}) \overset{H}{-} \int\limits_0^t \|a\|e^{-\|a\|s}dsB_1(\mathbf{0}) =$

$$= B_1(\mathbf{0}) \overset{H}{-} \left(1 - e^{-|a|t}\right) B_1(\mathbf{0}).$$

Hence $e^{-|a|t}B_1(\mathbf{0}) = B_1(\mathbf{0}) \overset{H}{-} \left(1 - e^{-|a|t}\right) B_1(\mathbf{0}) \implies e^{-|a|t}B_1(\mathbf{0}) + \left(1 - e^{-|a|t}\right) B_1(\mathbf{0}) = B_1(\mathbf{0}) \implies \left(e^{-|a|t} + 1 - e^{-|a|t}\right) B_1(\mathbf{0}) = B_1(\mathbf{0}) \implies B_1(\mathbf{0}) \equiv B_1(\mathbf{0})$

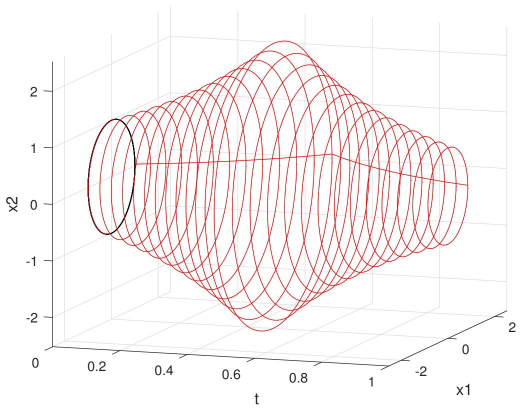

Figure 3: $a = 2$, $Y_1(t)$, $t \in [0,1]$

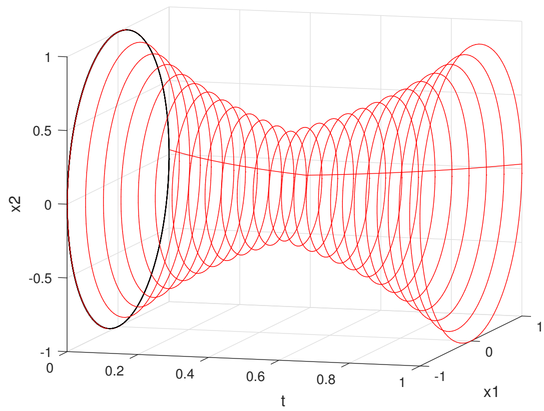

Figure 4: $a = 2$, $Y_2(t)$, $t \in [0,1]$

We also note that set-valued mappings

$$Y_1(t) = \begin{cases} B_{e^{|a|t}}(\mathbf{0}), & t \in [0, 0.5]; \\ B_{e^{|a|(1-t)}}(\mathbf{0}), & t \in [0.5, 1] \end{cases} \text{ and } Y_2(t) = \begin{cases} B_{e^{-|a|t}}(\mathbf{0}), & t \in [0, 0.5]; \\ B_{e^{|a|(t-1)}}(\mathbf{0}), & t \in [0.5, 1] \end{cases}$$

are the solutions of the Cauchy problem (3) for the differential equation with the PS-derivative and differential equation with the BG-derivative (see Figure 3 and Figure 4).

Also we note that the solution $Y_1(\cdot)$ is a solution of the integral equations

$$\text{(PS-derivative) } Y_1(t) = X_0 + \int_0^{m(t)} AY_1(s)ds \overset{H}{-} \theta(t-0.5)\int_{0.5}^{l(t)} AY_1(s)ds,$$

and

$$\text{(BG-derivative) } Y_1(t) = X_0 + \int_0^{m(t)} AY_1(s)ds \overset{H}{-} (-1)\theta(t-0.5)\int_{0.5}^{l(t)} AY_1(s)ds,$$

where $t \in [0,1]$, $m(t) = \min\{t,\, 0.5\}, l(t) = \max\{t,\, 0.5\}, \theta(t)$ is the Heaviside function.

Similarly, the solution $Y_2(\cdot)$ is a solution of the integral equations

$$\text{(PS-derivative) } Y_2(t) = X_0 \overset{H}{-} \int_0^{m(t)} AY_2(s)ds + \theta(t-0.5)\int_{0.5}^{l(t)} AY_2(s)ds,$$

and

$$\text{(BG-derivative) } Y_2(t) = X_0 \overset{H}{-} (-1)\int_0^{m(t)} AY_2(s)ds + \theta(t-0.5)\int_{0.5}^{l(t)} AY_2(s)ds.$$

Obviously, in this example, there are infinitely many such solutions. These solutions will be called mixed solutions. For these mixed solutions $Y(\cdot)$, the diameter function $\mathrm{diam}(Y(\cdot))$ is not increasing or decreasing over the entire interval.

It is obvious that if $a = 0$, then the Cauchy problem (3) for the differential equation with Hukuhara derivative, the differential equation with PS-derivative and the differential equation with BG-derivative will have the unique solution $X(t) = B_1(\mathbf{0})$.

Remark 13. Note also that in this example, the shape of the section of the solutions corresponds to the shape of the original set, and the dimensions of the section of the solution do not depend on the parameter ϕ.

However, if $A = \begin{pmatrix} a\cos(\phi) & -b\sin(\phi) \\ b\sin(\phi) & a\cos(\phi) \end{pmatrix}$, $a, b \in \mathbb{R}$ and $\phi \in [0, 2\pi)$ are numbers, then the singular values of the matrix A are equal to each other for any a, b and ϕ and $\sigma_1(a,b,\phi) = \sigma_2(a,b,\phi) = \sqrt{a^2\cos^2(\phi) + b^2 sin^2(\phi)}$. Therefore, the matrix A can be written as $\sigma_1(a,b,\phi)R(\psi)$, where $\psi = \arccos\left(\frac{a\,\cos(\phi)}{\sigma_1(a,b,\phi)}\right)$. Obviously, the set-valued mappings $X_1(t) = e^{\sigma_1(a,b,\phi)\,t}B_1(\mathbf{0})$ and $X_2(t) = e^{-\sigma_1(a,b,\phi)\,t}B_1(\mathbf{0})$ are basic solutions of the Cauchy problem (3) for the differential equation with the PS-derivative and the Cauchy problem (3) for the differential equation with the BG-derivative. In this case, the size of the section of the solutions will depend on a, b and ϕ.

Remark 14. In this example, we have

1) $X_0 = (-1)X_0 \implies X_0 \overset{H}{-} (-1)X_0 = \{\mathbf{0}\}$ [19];
2) as $X_0 = (-1)X_0$ and $AX_0 = \sigma_1 X_0$, then there exists an $\alpha > 0$ such that H-differences $X_0 \overset{H}{-} \alpha AX_0$ and $X_0 \overset{H}{-} \alpha(-1)AX_0$ exist [19].

Also, we note that solutions of differential equations with PS-derivative will be solutions of the differential equation with BG-derivative and vice versa.

Remark 15 ([19]). If H-difference $X_0 \overset{H}{-} (-1)X_0$ exists, then

1) if H-difference $X_0 \overset{H}{-} \alpha A X_0$ exists, then H-difference $X_0 \overset{H}{-} \alpha(-1)AX_0$ exists and vice versa;
2) if H-difference $X_0 \overset{H}{-} \alpha A X_0$ does not exist, then H-difference $X_0 \overset{H}{-} \alpha(-1)AX_0$ does not exist and vice versa.

Next, will look at an example when $X_0 \overset{H}{-} (-1)X_0 = \{\mathbf{g}\}$, but $\mathbf{g} \neq \mathbf{0}$ ($X_0 \neq (-1)X_0$) and there exists an $\alpha > 0$ such that H-differences $X_0 \overset{H}{-} \alpha A X_0$ and $X_0 \overset{H}{-} \alpha(-1)AX_0$ exist.

Example 2. Let $A = \begin{pmatrix} 1 & -1/2 \\ 1/2 & 1 \end{pmatrix}$, $\mathbf{b} = (1,1)^T$ and

$$DX(t) = AX(t),\ X(0) = B_1(\mathbf{b}),\ t \in [0,1], \tag{4}$$

It is easy to check that the singular values σ_1, σ_2 of the matrix A are equal to $\sqrt{1.25} \approx 1.118$ and the matrix A can be represented as $\begin{pmatrix} 1.118\cos(\phi) & -1.118\sin(\phi) \\ 1.118\sin(\phi) & 1.118\cos(\phi) \end{pmatrix}$, where $\phi \approx 26.6^\circ$. Hence, the system (4) has two basic solutions (see Example 1).

The set-valued mappings $X_1^{ps}(\cdot)$ and $X_2^{ps}(\cdot)$ are basic solutions of the Cauchy problem (4) for the differential equation with PS-derivative (see Figure 5 and Figure 6).

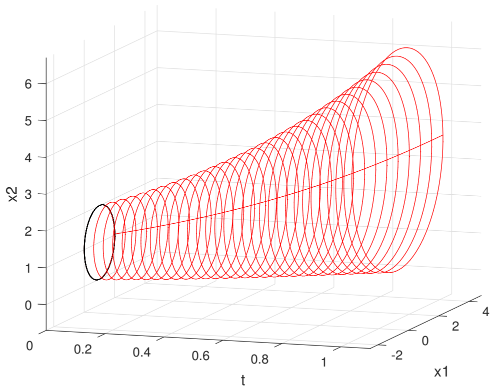

Figure 5:
$X_1^{ps}(t) = B_{e^{1.118t}}(\mathbf{e^{1.118t}}),\ t \in [0,1]$

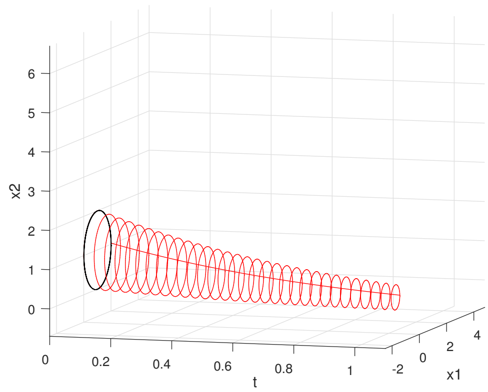

Figure 6:
$X_2^{ps}(t) = B_{e^{-1.118t}}(\mathbf{e^{-1.118t}}),\ t \in [0,1]$

The set-valued mappings $X_1^{bg}(\cdot)$ and $X_2^{bg}(\cdot)$ are basic solutions of the Cauchy problem (4) for the differential equation with BG-derivative (see Figure 7 and Figure 8).

Obviously, in this example, we have two different second basic solutions (see Figure 6 and Figure 8). This is true because $X_0 = B_1(\mathbf{b}) \neq (-1)X_0 = (-1)B_1(\mathbf{b}) = B_1((-1)\mathbf{b})$, but the H-difference $X_0 \overset{H}{-} (-1)X_0$ exists and is equal to $\{2\mathbf{b}\}$.

Remark 16. In this case, the Cauchy problem (1) for the differential equation with PS-derivative and the Cauchy problem (1) for the differential equation with BG-derivative have two basic solutions and their second basic solutions will be different.

Now we consider the following example, when H-differences

$$X_0 \overset{H}{-} \alpha A X_0 \quad \text{and} \quad X_0 \overset{H}{-} \alpha(-1)AX_0$$

do not exist for all $\alpha > 0$.

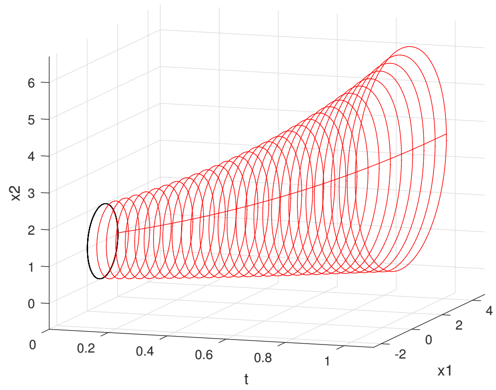

Figure 7:
$X_1^{bg}(t) = B_{e^{1.118t}}(\mathbf{e^{1.118t}})$, $t \in [0,1]$

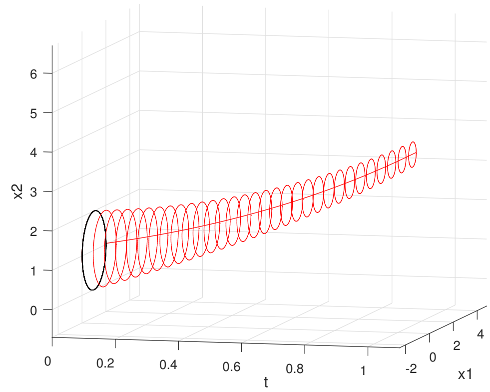

Figure 8:
$X_2^{bg}(t) = B_{e^{-1.118t}}(\mathbf{e^{1.118t}})$, $t \in [0,1]$

Example 3. Let $A = \begin{pmatrix} 4/7 & 2/3 \\ 2/3 & 6/5 \end{pmatrix}$ and

$$DX(t) = AX(t),\ X(0) = B_1(\mathbf{0}),\ t \in [0,1]. \tag{5}$$

Obviously, the matrix A can be represented in the form

$$\begin{pmatrix} \alpha_{11}(\phi)\cos(\phi) & -\alpha_{12}(\phi)\sin(\phi) \\ \alpha_{21}(\phi)\sin(\phi) & \alpha_{22}(\phi)\cos(\phi) \end{pmatrix},$$

where $\phi \in (0, 2\pi)$ and $\phi \neq \frac{k\pi}{2}$, $k = 1,2,3$. For example, if $\phi = \frac{\pi}{4}$ then $\alpha_{11}(\phi) = \frac{4}{7\cos(\phi)} = \frac{4\sqrt{2}}{7}$, $\alpha_{12}(\phi) = -\frac{2}{3\sin(\phi)} = -\frac{2\sqrt{2}}{3}$, $\alpha_{21}(\phi) = \frac{2}{3\sin(\phi)} = \frac{2\sqrt{2}}{3}$, $\alpha_{22}(\phi) = \frac{6}{5\cos(\phi)} = \frac{6\sqrt{2}}{5}$. However, for any admissible ϕ, we will not be able to get one of those options considered in Example 1, i.e. $\alpha_{11}(\phi) = \alpha_{12}(\phi) = \alpha_{21}(\phi) = \alpha_{22}(\phi)$ or $\alpha_{11}(\phi) = \alpha_{22}(\phi)$ and $\alpha_{12}(\phi) = \alpha_{21}(\phi)$.

By Proposition 1, the singular values of the matrix A are not equal to each other ($\sigma_1 = 1.62$, $\sigma_2 = 0.15$). From Theorem 1, the matrix A can always be decomposed in the form $U\Sigma V^T$. Just as matrix A is symmetric positive definite, it can be represented in the form $U\Sigma U^T$ [5, 12]:

$$\begin{pmatrix} 4/7 & 2/3 \\ 2/3 & 6/5 \end{pmatrix} = \begin{pmatrix} -0.54 & -0.84 \\ -0.84 & 0.54 \end{pmatrix} \begin{pmatrix} 1.62 & 0 \\ 0 & 0.15 \end{pmatrix} \begin{pmatrix} -0.54 & -0.84 \\ -0.84 & 0.54 \end{pmatrix}.$$

Obviously, $U = U^T = R(\psi)R_y$, where $R_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is the y-axis reflection matrix, $\psi \approx 57.6°$.

Hence $U^T B_1(\mathbf{0}) = (R(\psi)R_y)B_1(\mathbf{0}) = R(\psi)(R_y B_1(\mathbf{0})) = R(\psi)B_1(\mathbf{0}) = B_1(\mathbf{0})$. Further,

$$\Sigma B_1(\mathbf{0}) = Y = \left\{ (x_1, x_2)^T \in R^2 \ :\ \left(\frac{x_1}{\sigma_1}\right)^2 + \left(\frac{x_2}{\sigma_2}\right)^2 \leq 1 \right\}$$

is an ellipse.

As $U = R(\psi)R_y$ and $R_y Y = Y$ then $R(\psi)Y = Z$ is a rotated by ψ ellipse in which half-diagonals are equal to σ_1 and σ_2, i.e.

$$Z = \left\{ (x_1, x_2)^T \in \mathbb{R}^2 : \frac{(x_1\cos(\psi) + x_2\sin(\psi))^2}{\sigma_1^2} + \frac{(-x_1\sin(\psi) + x_2\cos(\psi))^2}{\sigma_2^2} \leq 1 \right\}.$$

Obviously, the H-differences $B_1(\mathbf{0})\overset{H}{-}\alpha AB_1(\mathbf{0})$ and $B_1(\mathbf{0})\overset{H}{-}(-1)\alpha AB_1(\mathbf{0})$ for all $\alpha > 0$ do not exist (the Hukuhara difference between the ball and the ellipsoid does not exist [1]), then the problem (5) for the differential equation with PS-derivative and the problem (5) for the differential equation with BG-derivative do not have a second basic solution, i.e. this the Cauchy problem will have only one basic solution $X(\cdot)$, i.e. the first basic solution $X(t) = \exp(At)B_1(\mathbf{0})$:

$$D_H(\exp(At)B_1(\mathbf{0})) = A\exp(At)B_1(\mathbf{0}) \Longrightarrow \frac{d(\exp(At))}{dt}B_1(\mathbf{0}) = A\exp(At)B_1(\mathbf{0})$$
$$\Longrightarrow A\exp(At)B_1(\mathbf{0}) \equiv A\exp(At)B_1(\mathbf{0}),\ X(0) = \exp(0\cdot A)B_1(\mathbf{0}) = B_1(\mathbf{0}).$$

Because $A = U\Sigma U^T$, then $\exp(At)B_1(\mathbf{0}) = \exp(U\Sigma U^T t)B_1(\mathbf{0}) = U\exp(\Sigma t)U^T B_1(\mathbf{0}) = R(\psi)\exp(\Sigma t)B_1(\mathbf{0})$.

Consequently, the section of the first basic solution $X(t)$ at each moment of time t will be a rotated by ψ ellipse in which the half-diagonals are equal to $e^{\sigma_1 t}$ and $e^{\sigma_2 t}$, i.e.

$$X(t) = R(\psi)\exp(t\Sigma)B_1(0) =$$
$$= \left\{(x_1,x_2)^T \in \mathbb{R}^2 \,:\, \frac{(x_1\cos(\psi)+x_2\sin(\psi))^2}{e^{2\sigma_1 t}} + \frac{(-x_1\sin(\psi)+x_2\cos(\psi))^2}{e^{2\sigma_2 t}} \le 1\right\}$$

(see Figure 9 and Figure 10).

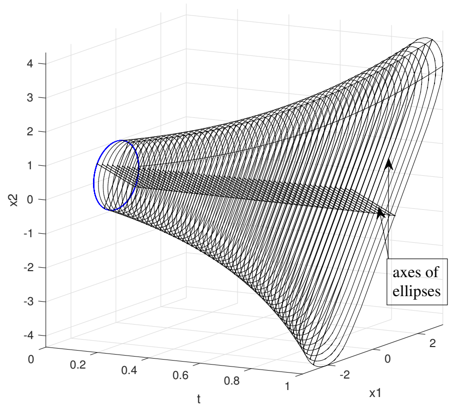

Figure 9: $X(t),\ t \in [0,1]$

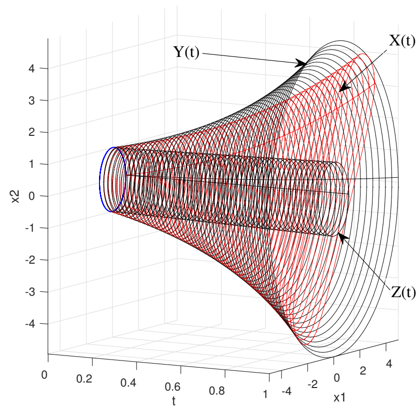

Figure 10: $Z(t) = B_{e^{0,15\,t}}(\mathbf{0}),\ Y(t) = B_{e^{1,62\,t}}(\mathbf{0}),$ $X(t), \quad t \in [0,1]$

Also note that this Cauchy problem has infinitely many mixed solutions. For example, the solution $Y_1(\cdot)$ such that the function $\mathrm{diam}(Y_1(\cdot))$ increases on $(0, 0.75)$ and decreases on $(0.75, 1)$ or the solution $Y_2(\cdot)$ such that the function $\mathrm{diam}(Y_2(\cdot))$ increases on $(0, 0.5)$ and $(0.75, 1)$ and decreases on $(0.5, 0.75)$ (see Figures 11 and 12).

Remark 17. If the matrix A is not symmetric and has different singular values, then it is obvious that additional calculations using matrix analysis will be required to find the rotation angle of the ellipse. We also note that obtaining analytical solutions is not the purpose of this paper.

Later in this paper we will consider only the basic solutions. The following remarks are obvious.

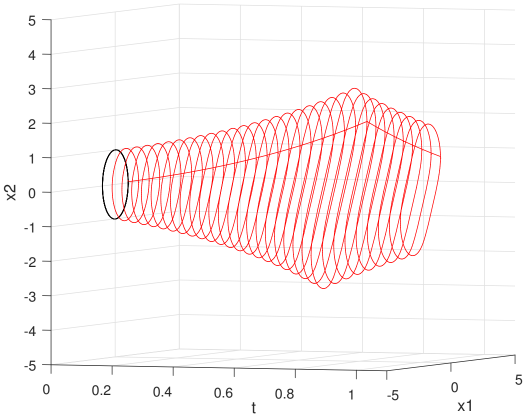

Figure 11: $Y_1(t)$, $t \in [0,1]$

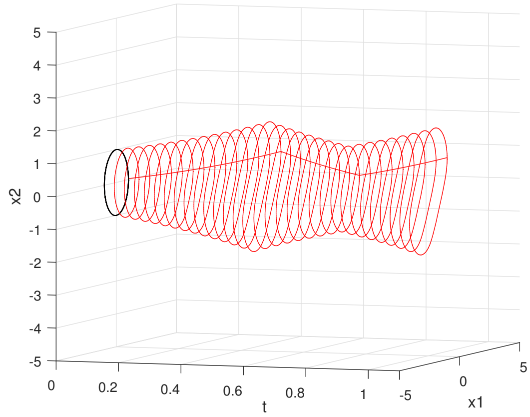

Figure 12: $Y_2(t)$ $t \in [0,1]$

Remark 18. If X_0 is a ball and all singular values of the matrix A are equal to each other, then the Cauchy problem for the differential equations with PS-derivative and BG-derivative have two basic solutions.

Remark 19. If X_0 is a ball and at least two singular values of the matrix A are not equal to each other, then the Cauchy problem for the differential equations with PS-derivative and BG-derivative have only the first basic solution.

Next, consider an example when the Hukuhara difference $X_0 \overset{H}{-} (-1)X_0$ does not exist.

Example 4. Let

$$DX(t) = AX(t),\ X(0) = K,\ t \in [0,1], \tag{6}$$

where $A \in \mathbb{R}^{2\times 2}$ such that $A = aR(\phi)$, $a \in \mathbb{R}$ and $\phi \in [0, 2\pi)$ are numbers, K is the triangle with vertices $(-1;0)^T$, $(1;0)^T$ and $(0;1)^T$.

It is obvious that $K \neq (-1)K$ and the Hukuhara difference $K \overset{H}{-} (-1)K$ does not exist.

If $a > 0$ and $\phi = 0$ or $a < 0$ and $\phi = \pi$, then $AK = |a|K$ and H-differences $K \overset{H}{-} \alpha AK$ for all $\alpha \in (0, |a|^{-1})$ exist and the Cauchy problem (6) for the differential equation with PS-derivative has two basic solutions $X_1(\cdot)$ and $X_2(\cdot)$ (see Figure 13 and Figure 14).

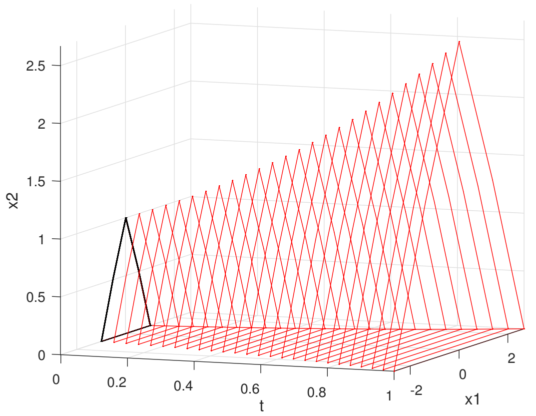

Figure 13: $a = 1$, $\phi = 0$, $X_1(t)$, $t \in [0,1]$

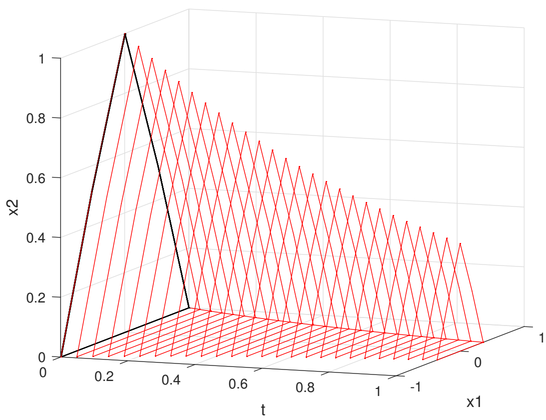

Figure 14: $a = 1$, $\phi = 0$, $X_2(t)$, $t \in [0,1]$

However, in this case, the Hukuhara difference $K \overset{H}{-} (-1)\alpha AK$ for all $\alpha > 0$ does not exist and the Cauchy problem (6) for the differential equation with BG-derivative has only the first basic solution $X(\cdot)$, which coincides with the first basic solution $X_1(\cdot)$ of the Cauchy problem (6) for the differential equation with PS-derivative.

If $a < 0$ and $\phi = 0$ or $a > 0$ and $\phi = \pi$, then $AK = -|a|K$ and H-differences $K \overset{H}{-} (-1)\alpha AK$ for all $\alpha \in (0, |a|^{-1})$ exist and the Cauchy problem (6) for the differential equation with BG-derivative has two basic solutions $X_1(\cdot)$ and $X_2(\cdot)$ (see Figure 15 and Figure 16).

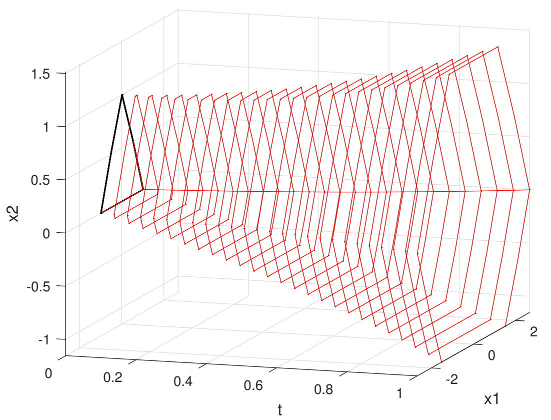

Figure 15: $a = 1,\ \phi = \pi,\ \ X_1(t),\ t \in [0, 1]$

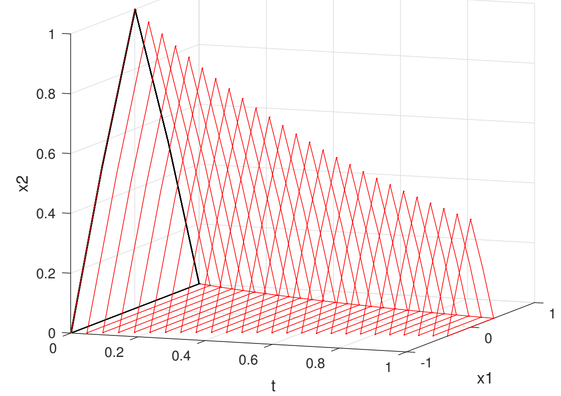

Figure 16: $a = 1,\ \phi = \pi,\ \ X_2(t),\ \ t \in [0, 1]$

However, in this case, the Hukuhara difference $K \overset{H}{-} \alpha AK$ for all $\alpha > 0$ does not exist and the Cauchy problem (6) for the differential equation with PS-derivative has only the first basic solution $X(\cdot)$, which coincides with the first basic solution $X_1(\cdot)$ of the Cauchy problem (6) for the differential equation with BG-derivative.

Also note that if $\phi \neq k\pi$, $k = 0, 1$, then AK is the triangle rotated at an angle ϕ. Hence, H-differences $K \overset{H}{-} \alpha AK$ and $K \overset{H}{-} (-1)\alpha AK$ for all $\alpha > 0$ do not exist (triangles K and AK are not homothetic) and the Cauchy problem (6) for the differential equation with PS-derivative and the Cauchy problem (6) for the differential equation with BG-derivative have only one solution $X(\cdot)$, which coincides with the solution of the Cauchy problem (6) for the differential equation with the Hukuhara derivative (see Figure 17).

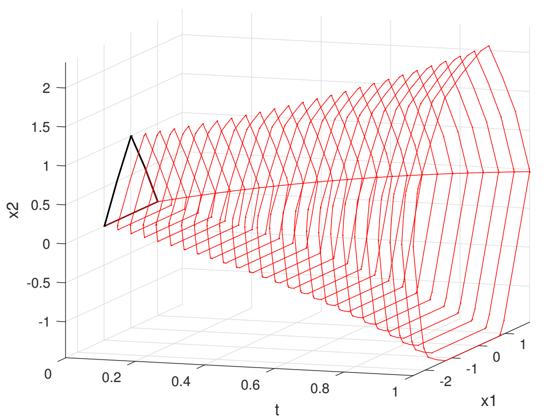

Figure 17: $a = 1,\ \phi = \frac{\pi}{3},\ \ X(t),\ t \in [0, 1]$

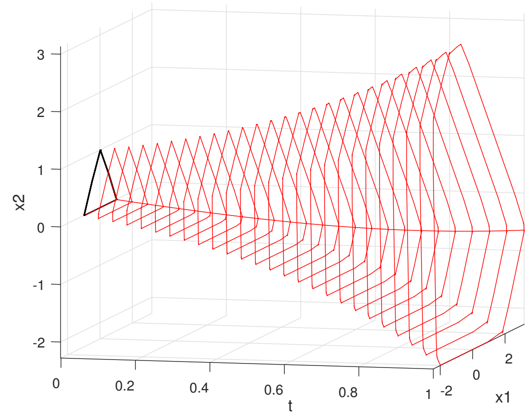

Figure 18: $X(t)$, $t \in [0, 1]$, when
$$A = \begin{pmatrix} -1 & -0.89 \\ 0.93 & -0.98 \end{pmatrix},\ \sigma_1 = 1.37,\ \sigma_2 = 1.32$$

Remark 20. In this case, the existence of two basic solutions guarantees the condition of coincidence of singular numbers, as well as a certain angle of rotation, that is, only if the matrix has the following form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, then if $a > 0$, then the Cauchy problem (6) for the differential equation with PS-derivative has two basic solutions, if $a < 0$, then the Cauchy problem (6) for the differential equation with BG-derivative has two basic solutions.

We also note that if the matrix $A \neq aR(\phi)$ and has different singular values, then H-differences $K \overset{H}{-} \alpha AK$ and $K \overset{H}{-} (-1)\alpha AK$ for all $\alpha > 0$ do not exist (triangles K and AK are not homothetic), i.e. system (6) has only the first basic solution, regardless of the type of the derivative (see example Figure 18).

Based on all stated above, we can make the following proposition.

Proposition 2. *For system (1) the following statements are true:*

1) if H-difference $X_0 \overset{H}{-} (-1)X_0 = \{\mathbf{0}\}$ *and there exists an* $\alpha > 0$ *such that H-difference* $X_0 \overset{H}{-} \alpha AX_0$ *exists, then the Cauchy problem (1) for the differential equation with PS-derivative and the Cauchy problem (1) for the differential equation with BG-derivative have two basic solutions and they are equivalent;*

2) if H-difference $X_0 \overset{H}{-} (-1)X_0 = \{\mathbf{g}\}$, $\mathbf{g} \neq \mathbf{0}$ *and there exists an* $\alpha > 0$ *such that H-difference* $X_0 \overset{H}{-} \alpha AX_0$ *exists, then the Cauchy problem (1) for the differential equation with PS-derivative and the Cauchy problem (1) for the differential equation with BG-derivative have two basic solutions and their second basic solutions will be different.*

3) if the H-difference $X_0 \overset{H}{-} (-1)X_0$ *does not exist,*

a) but there exists an $\alpha > 0$ *such that H-difference* $X_0 \overset{H}{-} \alpha AX_0$ *exists, then there are two basic solutions of the Cauchy problem (1) for the differential equation with PS-derivative and the first basic solution of the Cauchy problem (1) for the differential equation with BG-derivative;*

b) but there exists an $\alpha > 0$ *such that H-difference* $X_0 \overset{H}{-} (-1)\alpha AX_0$ *exists, then there are two basic solutions of the Cauchy problem (1) for the differential equation with BG-derivative and the first basic solution of the Cauchy problem (1) for the differential equation with PS-derivative;*

that is, in this case, the second base solution for the Cauchy problem for the differential equations with PS-derivative and the Cauchy problem for the differential equations with BG-derivative cannot exist simultaneously.

Remark 21. It is easy to see that in order to check the existence of two basic solutions, it is necessary to check the existence of the following Hukuhara differences:

$$X_0 \overset{H}{-} (-1)X_0,\; X_0 \overset{H}{-} \alpha AX_0,\; X_0 \overset{H}{-} (-1)\alpha AX_0.$$

Also we note that to check the existence of the second and third differences, one can use the properties of the singular values of the matrix A. However, we note that the presence of different singular values (not all singular values are equal) does not guarantee the existence of only the first basic solution, i.e. the absence of the second basic solution.

Let us show this in the following example.

Example 5. Let

$$DX(t) = AX(t),\; X(0) = C,\; t \in [0, 1], \tag{7}$$

where $A = \begin{pmatrix} a_{11}\cos(\phi) & -a_{12}\sin(\phi) \\ a_{21}\sin(\phi) & a_{22}\cos(\phi) \end{pmatrix}$, $a_{ij} \in \mathbb{R}\,(i,j = 1,2)$ and $\phi \in [0, 2\pi)$ are numbers, $C = \{x \in \mathbb{R}^2 \,:\, |x_i| \le 1,\ i = 1,2\}$ is a square.

It is obvious that $C = (-1)C$. Next, we will consider some cases for the matrix A.

A. Let $a_{ij} = a \neq 0$, $i, j = 1,2$, i.e. $A = aR(\phi)$. The singular values of the matrix A are equal to each other for any a and ϕ, i.e. $\sigma_1 = \sigma_2 = |a|$.

If $\phi = \frac{\pi k}{2}$, $k = 0,1,2,3$, then $AC = |a|C$ and H-differences $C \overset{H}{-} \alpha AC$ and $C \overset{H}{-} (-1)\alpha AC$ for all $\alpha \in (0, |a|^{-1})$ exist and the Cauchy problem (7) for the differential equation with PS-derivative and the Cauchy problem (7) for the differential equation with BG-derivative have two basic solutions $X_1(\cdot)$ and $X_2(\cdot)$ (see Figure 19 and Figure 20).

If $\phi \neq \frac{\pi k}{2}$, $k = 0,1,2,3$, then AC is a square rotated at an angle ϕ. Hence, H-differences $C \overset{H}{-} \alpha AC$ and $C \overset{H}{-} (-1)\alpha AC$ for all $\alpha > 0$ do not exist and the Cauchy problem (7) for the differential equation with PS-derivative and the Cauchy problem (7) for the differential equation with BG-derivative have only first basic solution $X(\cdot)$ such that $X(t) = C + X_{11}(t)$, where $X_{11}(t) = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1\cos(\phi) + x_2\sin(\phi)| \le e^{|a|t} - 1, |-x_1\sin(\phi) + x_2\cos(\phi)| \le e^{|a|t} - 1\}$ (see Figure 21).

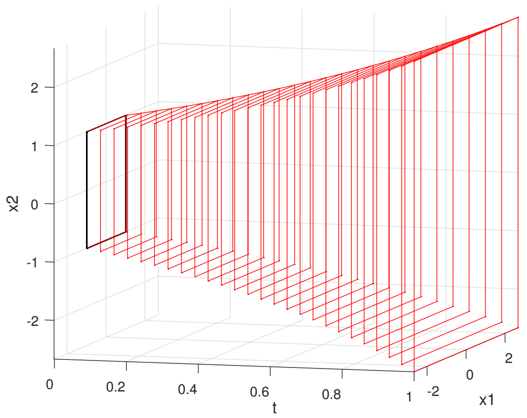

Figure 19: $a = 1$, $\phi = \frac{\pi}{2}$, $t \in [0,1]$,
$X_1(t) = \{x \in R^2 \,|\, |x_i| \le e^{|a|t},\ i = 1,2\}$

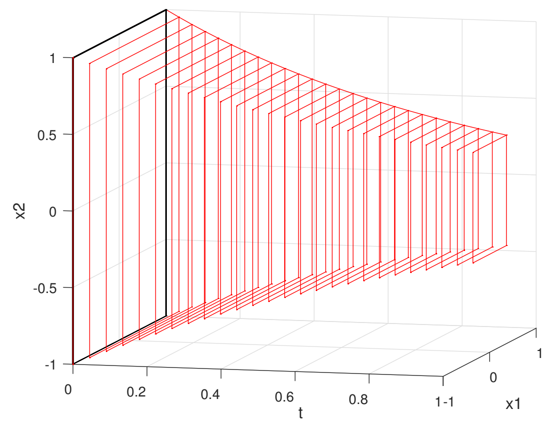

Figure 20: $a = 1$, $\phi = \frac{\pi}{2}$, $t \in [0,1]$,
$X_2(t) = \{x \in R^2 \,|\, |x_i| \le e^{-|a|t},\ i = 1,2\}$

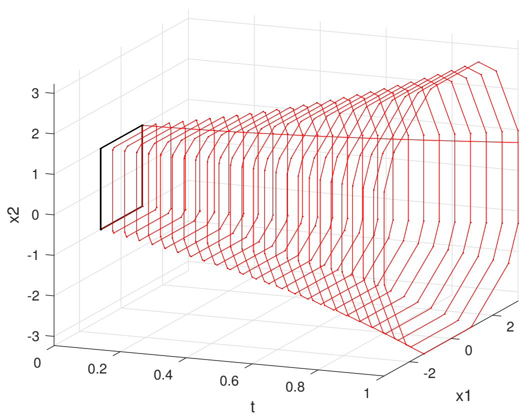

Figure 21: $a = 1$, $\phi = \frac{\pi}{6}$, $X(t)$, $t \in [0,1]$

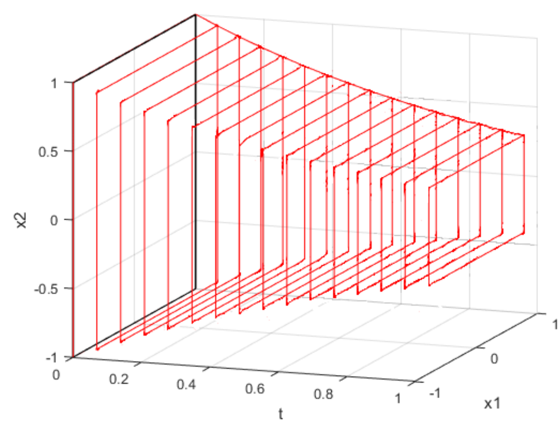

Figure 22: $a = \frac{1}{4}$, $b = 1$, $\phi = \pi$, $X_2(t)$, $t \in [0,1]$

B. Let $a_{11} = a_{22} = a$, $a_{21} = a_{12} = b$ and $|a| \neq |b|$. The singular values of the matrix A

are equal to each other for any a and ϕ, i.e. $\sigma_1 = \sigma_2 = \sqrt{a^2\cos^2(\phi) + b^2\sin^2(\phi)}$. Obviously, if $\phi = \frac{\pi k}{2},\ k = 0, 2$, then $AC = |a|C$ and if $\phi = \frac{\pi k}{2},\ k = 1, 3$, then $AC = |b|C$. Consequently, H-differences $C \overset{H}{-} \alpha AC$ and $C \overset{H}{-} (-1)\alpha AC$ for all $\alpha \in (0, \min\{|a|^{-1}, |b|^{-1}\})$ exist and the Cauchy problem (7) for the differential equation with PS-derivative and the Cauchy problem (7) for the differential equation with BG-derivative have two basic solutions $X_1(\cdot)$ and $X_2(\cdot)$.

If $\phi \neq \frac{\pi k}{2},\ k = 0, 1, 2, 3$, then AC is a square rotated at an angle $\psi(a, b, \phi) \neq \frac{\pi k}{2},\ k = 0, 1, 2, 3$. Hence, H-differences $C \overset{H}{-} \alpha AC$ and $C \overset{H}{-} (-1)\alpha AC$ for all $\alpha > 0$ do not exist and the Cauchy problem (7) for the differential equation with PS-derivative and the Cauchy problem (7) for the differential equation with BG-derivative have only one basic solution $X(\cdot)$. In this case, the graphs of the solutions of system (7) are similar to those given in the case A (see Figure 19, Figure 20 and Figure 21).

C. Let $a_{11} = a_{12} = a,\ a_{21} = a_{22} = b$ and $|a| \neq |b|$. In this case, the singular values of the matrix A are equal to $\sigma_1 = \max\{|a|, |b|\},\ \sigma_2 = \min\{|a|, |b|\}$.

If $\phi = \frac{\pi k}{2},\ k = 0, 1, 2, 3$, then AC is the rectangle with sides $2|a|$ and $2|b|$. Consequently, H-differences $C \overset{H}{-} \alpha AC$ and $C \overset{H}{-} (-1)\alpha AC$ for all $\alpha \in (0, (\max\{|a|, |b|\})^{-1})$ exist (H-difference of a square and a rectangle is a rectangle) and the Cauchy problem (7) for the differential equation with PS-derivative and the Cauchy problem (7) for the differential equation with BG-derivative have two basic solutions $X_1(\cdot)$ and $X_2(\cdot)$. For example, if $a = \frac{1}{4}$, $b = 1, \phi = \pi$ then the second solution $X_2(\cdot)$ has the form $X(t) = \{\, (x_1, x_2) \in \mathbb{R}^2 \,:\, |x_1| \leq e^{-\frac{t}{4}},\ |x_2| \leq e^{-t}\}$ (see Figure 22).

If $\phi \neq \frac{\pi k}{2},\ k = 0, 1, 2, 3$, then AC is some quadrilateral that is not a rectangle. Hence, H-differences $C \overset{H}{-} \alpha AC$ and $C \overset{H}{-} (-1)\alpha AC$ for all $\alpha > 0$ do not exist and the Cauchy problem (7) for the differential equation with PS-derivative and the Cauchy problem (7) for the differential equation with BG-derivative have only one basic solution $X(\cdot)$.

D. Let $a_{ij},\ i, j = 1, 2$, be such that $|a_{ij}|,\ i, j = 1, 2$, are not equal to each other (all different). Obviously if $\phi = \frac{\pi k}{2},\ k = 0, 1, 2, 3$, then we get a case similar to case C. Also, if $\phi \neq \frac{\pi k}{2},\ k = 0, 1, 2, 3$, then AC is some quadrilateral that is not a rectangle. Hence, H-differences $C \overset{H}{-} \alpha AC$ and $C \overset{H}{-} (-1)\alpha AC$ for all $\alpha > 0$ do not exist and the Cauchy problem (7) for the differential equation with PS-derivative and the Cauchy problem (7) for the differential equation with BG-derivative have only one basic solution $X(\cdot)$.

Remark 22. Obviously, in this case, the properties of the singular numbers do not determine the number of basic solutions. Their number determines the angle of rotation. That is, if the matrix A has one of the following types $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ or $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$, then the system (7) has two basic solutions, where $a, b \in R \setminus 0$.

Conclusion. The paper shows that the Cauchy problem for the linear set-valued differential equation are significantly different from the Cauchy problem for an ordinary differential equation. In this Cauchy problem, the number of solutions may depend on the type (shape) of the initial set, the considered derivative, and the matrix on the right-hand side. The paper gives the conditions for the existence of two basic solutions for the case when $n \geq 2$. If $n = 1$, then the system turns into an interval differential equation and it always has two basic solutions. We also note that in [32–35] the authors considered another type of differential equations with PS-derivative, in which at most one solution can exist.

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