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I. B. Sheparovych

ON THE INTERPOLATION IN SOME CLASSES OF HOLOMORPHIC IN THE UNIT DISK FUNCTIONS

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There is considered an interpolation problem $f(\lambda_n) = b_n$ in the class of holomorphic in the unit disk $U(0;1) = \{z \in \mathbb{C}\colon |z| < 1\}$ functions of finite η-type, i.e such that

$$(\exists A > 0)(\forall z \in U(0;1))\colon \quad |f(z)| \le \exp\Big(A\eta\Big(\frac{A}{1-|z|}\Big)\Big),$$

where $\eta\colon [1;+\infty) \to [0;+\infty)$ is an increasing convex function with respect to $\ln t$ and $\ln t = o(\eta(t))$ $(t \to +\infty)$. There were received sufficient conditions of the interpolation problem solvability in terms of the counting functions

$$N(r) = \int_0^r \frac{(n(t)-1)^+}{t}dt \text{ and } N_{\lambda_n}(r) = \int_0^r \frac{(n_{\lambda_n}(t)-1)^+}{t}dt.$$

Earlier, in 2004, necessary conditions were obtained (Ukr. Math. J., **56** (2004), №3) in these terms. For the moderate growth of f (when the majorant $\eta = \psi$ satisfies the condition $\psi(2x) = O(\psi(x))$, $x \to +\infty$) that problem was solved in J. Math. Anal. Appl., **414** (2014), №1. In this paper, we remove any restrictions on the growth of η and construct an interpolation function f such that

$$(\exists A' > 0)(\forall z \in U(0;1))\colon \quad |f(z)| \le \exp\Big(\frac{A'}{(1-|z|)^{3/2}}\eta\Big(\frac{A'}{1-|z|}\Big)\Big).$$

1. Introduction and main results. Let $\eta\colon [1;+\infty) \to [0;+\infty)$ be an increasing function convex in $\ln t$ and $\ln t = o(\eta(t))$ as $t \to +\infty$; $\Lambda = (\lambda_n)$ be a non-zero sequence of different complex numbers such that $0 < |\lambda_n| \nearrow 1$;

$$N(r) = \int_0^r \frac{(n(t)-1)^+}{t}dt; \quad N_\zeta(r) = \int_0^r \frac{(n_\zeta(t)-1)^+}{t}dt,$$

where

$$n(t) := n(t;\Lambda) = \sum_{|\lambda_n|\le t} 1,\ n_\zeta(t) := n_\zeta(t;\Lambda) = \textstyle\sum_{|\lambda_n-\zeta|\le t} 1,$$

are the counting functions, $x^+ := \max\{0;x\}$.

Consider an interpolation problem

$$f(\lambda_n) = b_n \tag{1}$$

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in the class of holomorphic functions in the unit disk $U(0;1)=\{z\in\mathbb{C}\colon |z|<1\}$ such that

$$(\exists A>0)(\forall z\in U(0;1))\colon |f(z)|\leqslant \exp\Big\{A\eta\Big(\frac{A}{1-|z|}\Big)\Big\} \tag{2}$$

(which are called functions of finite η-type).

In 2004 prof. B.V. Vynnyt'skyi and author obtained [1] conditions of the solvability of the interpolation problem (1). The next assertion was proved.

Theorem A ([1]). *In order that for every sequence (b_n) of complex numbers with the property*

$$(\exists A_1>0)(\forall n\in\mathbb{N})\colon |b_n|\le \exp\left(A_1\eta\left(\frac{A_1}{1-|\lambda_n|}\right)\right) \tag{3}$$

interpolation problem (1) have a solution from the class (2) it is necessary that

$$(\exists A_2>0)(\forall r\in(0;1))\colon N(r)\le A_2\eta\left(\frac{A_2}{1-r}\right) \tag{4}$$

$$(\exists A_3>0)(\forall \delta\in(0;1))(\forall n)\colon N_{\lambda_n}(\delta(1-|\lambda_n|))\le A_3\eta\left(\frac{A_3}{1-|\lambda_n|}\right) \tag{5}$$

are it is sufficient that there exists a holomorphic function L from class (2) with simple zeros in (λ_n) and for some $A_4>0$ and for all $n\in\mathbb{N}$:

$$\ln|(1-|\lambda_n|)L'(\lambda_n))|\ge -A_4\eta\left(\frac{A_4}{1-|\lambda_n|}\right). \tag{6}$$

In this theorem, the necessary and sufficient conditions are different. The problem arises: to deduce sufficient conditions in terms of the counting functions $N(r)$ and $N_{\lambda_n}(r)$.

We note that interpolation problem in various subclasses of functions holomorphic in the unit disk was considered by a number of mathematicians. Carleson [12] and Jones [13] obtained results for bounded functions. K. Seip [14], A. Hartmann and X. Massaneda [15] studied interpolation sequences in the class of power functions. Spreading Seip's idea of construction peak function, A. Borichev, R. Dhuez and K. Kellay [16], solved an interpolation problem in the class of functions of arbitrary growth in the unit disc. But those results has lower limit of growth. In that case the majorant grows faster than $\ln\frac{1}{1-r}$ (more detailed analysis see in [2]).

In 2014, I. Chyzhykov and author [2] solved an interpolation problem (1) for the moderate growth of f. There was proved the next assertion

Theorem B ([2]). *Let $\psi\colon [1,+\infty)\to\mathbb{R}_+$ be a function such that $\psi(2x)=O(\psi(x))$, $x\to+\infty$. If*

$$(\forall n\in N)\colon N_{\lambda_n}\Big(\frac{1-|\lambda_n|}{2}\Big)\le\psi\Big(\frac{1}{1-|\lambda_n|}\Big),$$

then for any sequence (b_n) satisfying

$$\ln|b_n|\le\widetilde{\psi}\left(\frac{1}{1-|\lambda_n|}\right),\ n\in\mathbb{N},$$

there exists an analytic function f in $U(0;1)$ with the properties (1) and

$$\ln M(r,f)\le C_2\widetilde{\psi}\left(\frac{1}{1-r}\right),$$

where $C_2>0, \widetilde{\psi}(x)=\int_1^x\frac{\psi(t)}{t}dt$.

Theorem B solves the shortcomings of results from [16], but the majorant η grows not faster than $(1-r)^{-\rho}, \rho>0$.

The aim of the paper is finding conditions of the solvability problem (1) in the class of finite η-type for arbitrary growth of majorante η by the method of construction interpolation series as in [1, 2, 3]. For the goal we need to prove the existence such a finite η-type function L satisfying (6). We need to use the next results from [4].

Theorem C ([4]). *If conditions (4) and*

$$\Big|\frac{1}{2k}\sum_{r_1<|\lambda_\nu|\leqslant r_2}\frac{1}{\lambda_\nu^k}\Big|\leq\frac{A_2}{r_1^k}\eta\left(\frac{B_2}{1-r_1}\right)+\frac{A_2}{r_2^k}\max\left\{1;\frac{1}{k\ln\sigma_2}\right\}\eta\left(\frac{B_2}{1-\sigma_2r_2}\right) \tag{7}$$

are fulfilled for some $A_2>0$ and $B_2>0$ and all $k\in\mathbb{N}, r_1\in(0;1), r_2\in(r_1;1), \sigma_2\in(1;1/r_2)$, then there exists a holomorphic in $U(0;1)$ function L for which (λ_ν) is a sequence of zeros and

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\ln|L(re^{i\varphi})|\right|^2d\varphi\right)^{1/2}\leqslant A_3\eta\left(\frac{B_3}{1-\sigma r}\right)\left(1+\frac{6\sqrt{\ln 2}}{\sqrt{\ln\sigma}}\right) \tag{8}$$

for some positive constants $A_3>0$, $B_3>0$ and all $r\in(0;1),\sigma\in(1;1/r)$.

The goal of the paper is proving the next assertion.

Theorem 1. *If conditions (4), (5) hold then for each sequence (b_n) with the property (3) interpolation problem (1) has a solution in the class of holomorphic functions f satisfying*

$$(\forall z\in U(0;1))\colon|f(z)|\leq\exp\left(\frac{A'}{(1-|z|)^{3/2}}\eta\left(\frac{A'}{1-|z|}\right)\right) \tag{9}$$

for some $A'>0$.

At first, we will prove following propositions.

Theorem 2. *If condition (4) holds for the sequence (λ_ν), then there exists a function $\tilde L$ holomorphic in the unit disk, satisfying*

$$(\forall z\in U(0;1))\colon\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\ln|\widetilde L(re^{i\varphi})|\right|^2d\varphi\right)^{1/2}\leq\frac{A}{1-r}\eta\left(\frac{A}{1-r}\right), \tag{10}$$

for which (λ_ν) is a subsequence of zeros.

Theorem 3. *If condition (4) holds for the sequence (λ_ν), then there exists a function $\tilde L$ holomorphic in the unit disk, satisfying (9), for which (λ_ν) is zeros subsequence.*

2. Proof of the main results.

Proof of Theorem 2. We have to construct a sequence $\widetilde\Lambda:=(\widetilde\lambda_\nu)$ with the property (4) and (7). For that we use Beck's construction (see, for example, [5, 6]). It is an analogue of Miles's construction for entire functions [7]. We will repeat this process. We supplement the sequence Λ with a sequence $\Lambda^*:=(\lambda_\nu^*)$ as follows. Put $R_m=1-2^{-m}$, $m\in\mathbb{N}$. Let us take those λ_ν

from Λ lying in the annulus $\{z\colon R_m \le |z| < R_{m+1}\}$ for some fixed m. These are numbers $\lambda_\nu = |\lambda_\nu| e^{i\theta_\nu},\ 0 \le \theta_\nu < 2\pi,\ \nu \in \overline{1,p},\ p = n(R_{m+1}) - n(R_m)$. Define

$$s_m(\theta) := -2\sum_{\nu=1}^{p}\sum_{n=1}^{\infty}(R_{m-1}/|\lambda_\nu|)^n e^{in(\theta-\theta_\nu)};$$

$$h_m(\theta) := \mathrm{Re}\,(s_m(\theta))\,; \quad f_m(\theta) := h_m(\theta) + \frac{2pR_m}{1-R_m}.$$

Then $|h_m(\theta)| \le |s_m(\theta)| \le \frac{2pR_m}{1-R_m}$ and $0 \le f_m(\theta) \le \frac{4pR_m}{1-R_m}$. Let $l_m = [\frac{1}{2\pi}\int_0^{2\pi} f_m(\theta)d\theta]$, where $[a]$ is the entire part of a. For every k, $k \in \overline{1,l_m}$, by $(\theta_k^*),\ k \in \overline{1,l_m}$ we denote a monotonous sequence such that $\frac{1}{2\pi}\int_0^{\theta_k^*} f_m(\theta)d\theta = k$. We make a sequence $\Lambda^* = \bigcup_{m\ge 2}\Lambda_m^*$, where $\Lambda_m^* = \{R_{m-1}e^{i\theta^*{}_k}\colon k \in \overline{1,l_m}\}$. Then $\widetilde{\Lambda} = \Lambda \cup \Lambda^*$. (Note that if λ_ν coincides with λ_ν^* we will use them once). By construction

$$l_m \leqslant \frac{4pR_m}{1-R_m} = \frac{4R_m(n(R_{m+1}) - n(R_m))}{1-R_m},$$

and therefore (here $n^*(r) := n(r;\Lambda^*)$, $\tilde{n}(r) := n(r;\tilde{\Lambda})$),

$$n^*(r) \le \frac{4rn\left(\frac{r+1}{2}\right)}{1-r},$$

consequently $\tilde{n}(r) \le \left(1 + \frac{4r}{1-r}\right) n\left(\frac{r+1}{2}\right)$. Since for every $\sigma \in (1;1/s)$

$$N(\sigma s) \ge \int_s^{\sigma s}\frac{n(t)}{t}dt \ge n(s)\ln\sigma,$$

then $n(s) \le \frac{1}{\ln\sigma}N(\sigma s)$. So,

$$\tilde{n}(r) \le \frac{4}{(1-r)\ln\sigma}N\left(\sigma\frac{r+1}{2}\right) \le \frac{4A_2}{(1-r)\ln\sigma}\eta\left(\frac{A_2}{1-\sigma\frac{r+1}{2}}\right) \le \frac{A_2'}{(1-r)}\eta\left(\frac{A_2'}{1-r}\right) \tag{11}$$

if $\sigma = \frac{r+3}{2(1+r)}$.

In addition, the newly formed sequence satisfies condition (9). Really, let $r_1,\ r_2$ be arbitrary numbers from $(0;1)$ and $r_1 < r_2$. There exist such natural numbers p_1 and p_2, that

$$R_{p_1} \leqslant r_1 < R_{p_1+1} < \ldots < R_{p_2} \leqslant r_2 < R_{p_2+1},$$

where $R_k = 1 - \frac{1}{2^k}$, $k \in \mathbb{N}$. The annulus $U(0;r_1;r_2)$ is covering by annuli $U(0;R_m;R_{m+1})$, where $m \in [p_1;p_2]$. For some positive constant A_2, B_2 there holds

$$\left|\frac{1}{k}\sum_{r_1<|\tilde{\lambda}_\nu|\leqslant r_2}\frac{1}{\tilde{\lambda}_\nu^k}\right| \leqslant \frac{1}{k}\left|\sum_{r_1<|\lambda_\nu|\leqslant R_{p_1+2}}\frac{1}{\lambda_\nu^k}\right| + \frac{1}{k}\left|\sum_{R_{p_1+2}<|\lambda_\nu|\leqslant R_{p_1+3}}\frac{1}{\lambda_\nu^k} + \sum_{|\lambda_\nu^*|=R_{p_1+1}}\frac{1}{{\lambda_\nu^*}^k}\right| + \ldots +$$

$$+\frac{1}{k}\left|\sum_{R_{p_2-1}<|\lambda_\nu|\leqslant R_{p_2}}\frac{1}{\lambda_\nu^k} + \sum_{|\lambda_\nu^*|=R_{p_2-2}}\frac{1}{{\lambda_\nu^*}^k}\right| + \frac{1}{k}\left|\sum_{R_{p_2}<|\lambda_\nu|\leqslant r_2}\frac{1}{\lambda_\nu^k} + \sum_{|\lambda_\nu^*|=R_{p_2-1}}\frac{1}{{\lambda_\nu^*}^k}\right| +$$

$$+\frac{1}{k}\left|\sum_{|\lambda^*_\nu|=R_{p_2}}\frac{1}{\lambda_\nu^{*k}}\right|\le\frac{1}{k}\left|\sum_{r_1<|\lambda_\nu|\leqslant R_{p_1+2}}\frac{1}{\lambda_\nu^k}\right|+32\sum_{j=p_1+1}^{p_2-1}\frac{1}{R_j^k}+\frac{1}{k}\left|\sum_{|\lambda^*_\nu|=R_{p_2}}\frac{1}{\lambda_\nu^{*k}}\right|=$$

$$+\frac{1}{k}\left|\sum_{R_{p_2}<|\lambda_\nu|\leqslant r_2}\frac{1}{\lambda_\nu^k}\right|\le\frac{n(R_{p_1+2})}{kr_1{}^k}+32\sum_{j=p_1+1}^{p_2}\frac{1}{R_j^k}+\frac{n\left(r_2\right)-n(R_{p_2})}{kr_2{}^k},$$

because as proved in [5, 7],

$$\left|\sum_{R_{p_1+j}<|\lambda_\nu|\le R_{p_1+j+1}}\frac{1}{\lambda_\nu^k}+\sum_{|\lambda'_\nu|=R_{p_1+j-1}}\frac{1}{\lambda_\nu'^k}\right|\le\frac{32k}{R^k_{p_1+j-1}}.$$

Moreover, $R_{p_1+2}\le R_{p_2}\le r_2$, $R_{p_1+2}\le\frac{R_{p_1}+3}{4}\le\frac{r_1+3}{4}$, $R_{p_2+1}\le\frac{1+r_2}{2}$ and since function $\varphi(t)=\frac{1}{t^k}$ is decreasing, then

$$\sum_{j=p_1+1}^{p_2}\frac{1}{R_j^k}\le\frac{1}{R^k_{p_1+1}}+\int_{p_1+1}^{p_2}\frac{dt}{\left(1-\left(\frac{1}{2}\right)^t\right)^k}=\frac{1}{r_1^k}+\frac{1}{\ln 2}\int_{R_{p_1+1}}^{R_{p_2}}\frac{dt}{t^k(1-t)}\le$$

$$\le\frac{1}{r_1^k}+\frac{1}{r_1^k\ln 2}\int_{r_1}^{r_2}\frac{dt}{1-t}\le\frac{3}{r_1^k}\ln\frac{1}{1-r_1}.$$

If $\beta:=\frac{r_1}{r_2}\ge r_1$, then

$$1-\frac{r_1^{2k}}{r_2^{2k}}=(1-\beta)(1+\beta+\beta^2+...+\beta^{2k-1})\leqslant(1-\beta)2k\leqslant(1-r_1)2k.$$

Thereby from (4) and inequality $n(r)\le\frac{2}{1-r}N\left(\frac{r+1}{2}\right)$ we obtain

$$\left|\frac{1}{k}\sum_{r_1<|\tilde\lambda_\nu|\leqslant r_2}\frac{1}{\tilde\lambda_\nu^k}\right|\leqslant\frac{n\left(R_{p_1+2}\right)}{k}\left(\frac{1}{r_1^k}-\frac{1}{r_2^k}\right)+\frac{96}{r_1^k}\ln\frac{1}{1-r_1}+\frac{n(r_2)}{kr_2{}^k}\le$$

$$\le\frac{16}{r_1^k}N\left(\frac{r_1+7}{8}\right)+\frac{96}{r_1^k}\ln\frac{1}{1-r_1}+\frac{2}{k\ln\sigma_2 r_2^k}N\left(\sigma_2 r_2\right)\le$$

$$\le\frac{A_2}{r_1{}^k}\eta\left(\frac{B_2}{1-r_1}\right)+\frac{A_2}{r_2{}^k}\max\left\{1;\frac{1}{k\ln\sigma_2}\right\}\eta\left(\frac{B_2}{1-\sigma_2 r_2}\right).$$

The next we use Theorem C for the sequence $\widetilde\Lambda$. But it should be noted, that in the proof of theorem C (Theorem 2 in [4]) we, actually, use the condition $n(r)\le\frac{1}{\ln\sigma}\eta\left(\frac{A_2}{1-r}\right)$ (see Lemmas 3 – 5 from [4]). So, if the sequence $\widetilde\Lambda$ satisfies conditions (7) and (11), there exists a function $\tilde L$ holomorphic in the unit disk, satisfying (10). And (λ_ν) is a subsequence of zeros of function $\tilde L$. □

Proof of Theorem 3. There are well known relations for holomorphic function L in the unit disk (see, for example, [8], [9], p. 84) for some $A>0$ and all $R\in(r;1)$

$$\ln|L(re^{i\varphi})|\leqslant AM_p(R;L)(R-r)^{-1/p},$$

where

$$M_p(R;L) = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|\ln|L(Re^{i\varphi})|\right|^p d\varphi\right)^{1/p}.$$

So, for the function $\tilde{L}$ from Theorem 2, when $p=2$ and $R=\frac{r+1}{2}$ we have

$$\ln|\tilde{L}(re^{i\varphi})| \leqslant \frac{2A}{(1-r)^{3/2}}\eta\left(\frac{2A}{1-r}\right). \tag{12}$$

□

Now we prove the next assertion.

Lemma 1. *If condition (5) holds for Λ then it also holds for $\tilde{\Lambda}$.*

Proof. Since $n^*_{\lambda_n}(t) \leqslant c_1 n_{\lambda_n}\left(\frac{t+1}{2}\right)$ for some $c_1>0$, then for some $A_3'>0$, $\delta'\in(\delta;1)$ one has

$$\widetilde{N}_{\lambda_n}(\delta(1-|\lambda_n|)) = \int_0^{\delta(1-|\lambda_n|)} \frac{\widetilde{n}_{\lambda_n}(t)-1}{t}dt = \int_0^{\delta(1-|\lambda_n|)} \frac{n_{\lambda_n}(t)-1}{t}dt +$$
$$+\int_0^{\delta(1-|\lambda_n|)} \frac{(n^*_{\lambda_n}(t)-1)^+}{t}dt \le N_{\lambda_n}(\delta(1-|\lambda_n|)) + c_1 N_{\lambda_n}(\delta'(1-|\lambda_n|)) \le A_3'\eta\left(\frac{A_3'}{1-|\lambda_n|}\right).$$

□

Proof of Theorem 1. If condition (4) holds then from Theorem 2 we have an existence of the holomorphic function $\widetilde{L}$ in $U(0;1)$ from class (10) which has simple zeros in $(\widetilde{\lambda}_n)$. So, from the Jensen equality for $\widetilde{L}(\lambda_n+z)$ on a circle $\partial U(\lambda_n;\delta(1-|\lambda_n|))$ for every $\delta>0$ there follows

$$\int_0^{\delta(1-|\lambda_n|)} \frac{\tilde{n}_{\lambda_n}(t)-1}{t}dt = \frac{1}{2\pi}\int_0^{2\pi} \ln|\tilde{L}(\lambda_n+\delta(1-|\lambda_n|)e^{i\varphi})|d\varphi - \ln\left|\delta(1-|\lambda_n|)\tilde{L}'(\lambda_n)\right|.$$

Then by Lemma 1 and inequality

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\ln|\widetilde{L}(re^{i\varphi})|\right|d\varphi \le \left(\frac{1}{2\pi}\int_0^{2\pi}\left|\ln|\widetilde{L}(re^{i\varphi})|\right|^2 d\varphi\right)^{1/2} \le \frac{A}{1-r}\eta\left(\frac{A}{1-r}\right)$$

we have such an estimate

$$-\ln\left|(1-|\lambda_n|)\tilde{L}'(\lambda_n)\right| = \tilde{N}_{\lambda_n}\left(\delta(1-|\lambda_n|)\right) - \frac{1}{2\pi}\int_0^{2\pi}\ln|\tilde{L}(\lambda_n+\delta(1-|\lambda_n|)e^{i\varphi})|d\varphi \le$$
$$\le \frac{A_4}{1-|\lambda_n|}\eta\left(\frac{A_4}{1-|\lambda_n|}\right). \tag{13}$$

The next step of the proof is a construction of interpolation function. We will use methods from [1], [3], [2]. Then

$$f(z) = \sum_{k=1}^{+\infty} \frac{b_n}{\tilde{L}'(\lambda_n)}\frac{\tilde{L}(z)}{(z-\lambda_n)}\left(\frac{1-|\lambda_n|^2}{1-\bar{\lambda}_n z}\right)^{s_n-1},$$

where (s_n) is a certain sequence of natural numbers which will be choose later.

Since the function $t\eta(t)$ is convex in $\ln t$ then from [10] it folllows an existence of the entire function ψ such that ($c_0 \geq 4(A_2 + A_4 + A_1)$)

$$\ln M_\psi(t) = (1 + o(1))c_0 t\eta(c_0 t), \ t \to +\infty.$$

Since

$$\mu_\psi(t) \leq M_\psi(t) \leq (1 + 1/\varepsilon)\, \mu_\psi((1 + \varepsilon)t), \quad \varepsilon > 0,$$

where $\mu_\psi(t) = \max\{|\psi_n|t^n\}$ and $\psi_n = \psi^{(n)}(0)/n!$, then

$$\mu_\psi(t) \leq \exp(2c_0 t\eta(c_0 t)), \quad \mu_\psi(t) \geq \exp((c_0/4)t\eta(c_0 t/2)), \quad t \geq t_0. \tag{14}$$

Let us choose (s_n) such, that

$$\chi_{s_n}(\hat{\psi}) \leq \frac{1}{1 - |\lambda_n|} < \chi_{s_n+1}(\hat{\psi}),$$

(here $\hat{\psi}(z) = \sum_{n=0}^{+\infty} \hat{\psi}_n z^n$ is the Newton majorant of ψ, $\chi_n(\hat{\psi}) = |\hat{\psi}_{n-1}/\hat{\psi}_n|$). Then [11]

$$\mu_\psi\left((1 - |\lambda_n|)^{-1}\right) = \hat{\psi}_{s_n}(1 - |\lambda_n|)^{-s_n}, \quad \mu_\psi(t) \geq \hat{\psi}_{s_n} t^{s_n}. \tag{15}$$

So, from (3), (12)–(15) and relation (see, for example, [2])

$$\left|\frac{1 - \overline{\lambda_n} z}{\overline{\lambda_n}} \frac{\tilde{L}(z)}{z - \lambda_n}\right| \leq \exp\left(\frac{2A}{(1 - r)^{3/2}}\eta\left(\frac{A}{1 - r}\right)\right)$$

we get

$$\begin{aligned}
|f(z)| \leq \exp\left(\frac{2A}{(1 - r)^{3/2}}\eta\left(\frac{A}{1 - r}\right)\right)\sum_{n=1}^{\infty}\frac{|b_n|}{|(1 - |\lambda_n|)\tilde{L}'(\lambda_n)|}\frac{\hat{\psi}_{s_n}(2/(1 - r))^{s_n}}{\hat{\psi}_{s_n}(1 - |\lambda_n|)^{-s_n}} \leq \\
\leq \exp\left(\frac{2A}{(1 - r)^{3/2}}\eta\left(\frac{A}{1 - r}\right) + \frac{2c_0}{(1 - r)}\eta\left(\frac{2c_0}{1 - r}\right)\right) \times \\
\times \sum_{n=1}^{\infty}\exp\left(A_1\eta\left(\frac{A_1}{1 - |\lambda_n|}\right) + \frac{A_4}{1 - |\lambda_n|}\eta\left(\frac{A_4}{1 - |\lambda_n|}\right) - \frac{c_0/4}{1 - |\lambda_n|}\eta\left(\frac{c_0/2}{1 - |\lambda_n|}\right)\right) \leq \\
\leq \exp\left(\frac{A'}{(1 - r)^{3/2}}\eta\left(\frac{A'}{1 - r}\right)\right)\sum_{n=1}^{\infty}\exp\left(-\frac{A_2}{1 - |\lambda_n|}\eta\left(\frac{2A_2}{1 - |\lambda_n|}\right)\right) \leq \\
\leq \exp\left(\frac{A'}{(1 - r)^{3/2}}\eta\left(\frac{A'}{1 - r}\right)\right)\sum_{n=1}^{\infty}\exp(-n) \leq \exp\left(\frac{A'}{(1 - r)^{3/2}}\eta\left(\frac{A'}{1 - r}\right)\right).
\end{aligned}$$

Hence, the function f belongs to the class of holomorphic in the unit disk functions with property (9). □

Remark 3. We can see that a majorant of the class (9) differs from that's one of the class (2) by a multiplier $\frac{1}{(1-r)^{3/2}}$, but in the case when $\eta(t) = o(\eta(At))$, $(A > 1)$, as $t \to \infty$, a multiplier $\frac{1}{(1-r)^{3/2}}$ does not play a role in the product $\frac{1}{(1-r)^{3/2}}\eta(\frac{1}{1-r})$. So, Theorem 1 complements Theorem B in a certain sense.

Example 1. The function $\eta(t) = t^{\ln t}$ satisfies given conditions on the majorant η. In this case, class (12) coincides with the class (2).

Example 2. The function $\eta(t) = t^{\ln\ln t}$ satisfies given conditions too, and classes (12) and (2) coincide.

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Drohobych Ivan Franko State Pedagogical University
Drohobych, Ukraine
irsheparovych@gmail.com

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