УДК 517.5

R. Dmytryshyn, C. Cesarano, I.-A. Lutsiv, M. Dmytryshyn

NUMERICAL STABILITY OF THE BRANCHED CONTINUED FRACTION EXPANSION OF THE HORN'S HYPERGEOMETRIC FUNCTION H_4

R. Dmytryshyn, C. Cesarano, I.-A. Lutsiv, M. Dmytryshyn. *Numerical stability of the branched continued fraction expansion of Horn's hypergeometric function* H_4, Mat. Stud. **61** (2024), 51–60.

In this paper, we consider some numerical aspects of branched continued fractions as special families of functions to represent and expand analytical functions of several complex variables, including generalizations of hypergeometric functions. The backward recurrence algorithm is one of the basic tools of computation approximants of branched continued fractions. Like most recursive processes, it is susceptible to error growth. Each cycle of the recursive process not only generates its own rounding errors but also inherits the rounding errors committed in all the previous cycles. On the other hand, in general, branched continued fractions are a non-linear object of study (the sum of two fractional-linear mappings is not always a fractional-linear mapping). In this work, we are dealing with a confluent branched continued fraction, which is a continued fraction in its form. The essential difference here is that the approximants of the continued fraction are the so-called figure approximants of the branched continued fraction. An estimate of the relative rounding error, produced by the backward recurrence algorithm in calculating an nth approximant of the branched continued fraction expansion of Horn's hypergeometric function H4, is established. The derivation uses the methods of the theory of branched continued fractions, which are essential in developing convergence criteria. The numerical examples illustrate the numerical stability of the backward recurrence algorithm.

1. Introduction. Numerous studies show that branched continued fraction expansions provide a useful means for representing and extending of special functions, including generalized hypergeometric functions [3, 33], Appell's hypergeometric functions [11, 20, 25], Horn's hypergeometric functions [2, 4, 5, 6, 15], Lauricella–Saran's hypergeometric functions [1, 12, 24], and also some other functions [10, 17, 18, 29]. To render branched continued fractions more useful in computational, one needs to know more about their numerical stability, which is the main concern of this paper.

The backward recurrence algorithm for computing the nth approximant

$$f_n = 1 + a_{1,0} + \cfrac{a_{0,1}}{1 + a_{1,1} + \cfrac{a_{0,2}}{1 + a_{1,2} + \cfrac{a_{0,3}}{1 + \ddots + a_{1,n-2} + \cfrac{a_{0,n-1}}{1 + a_{1,n-1} + a_{0,n}}}}}$$

2020 *Mathematics Subject Classification:* 32A17, 33C65, 41A20, 65G50.
Keywords: branched continued fraction; Horn hypergeometric function; approximation by rational functions; roundoff error.
doi:10.30970/ms.61.1.51-60

of a branched continued fraction

$$1+a_{1,0}+\cfrac{a_{0,1}}{1+a_{1,1}+\cfrac{a_{0,2}}{1+a_{1,2}+\cfrac{a_{0,3}}{1+\ddots}}} \tag{1}$$

consists of setting $G_n^{(n)}=1$ and computing successively, from tail to head,

$$G_k^{(n)}=1+a_{1,k}+\frac{a_{0,k+1}}{G_{k+1}^{(n)}},\quad n-1\geq k\geq 0.$$

Thus,

$$f_n=G_0^{(n)}.$$

It should be noted that the branched continued fraction (1) is a confluent branched continued fraction, which is a continued fraction in its form. The essential difference here is that the approximants of the continued fraction are the so-called figure approximants of the branched continued fraction (see, [8, p. 18]).

Branched continued fractions of the structure (1) appeared thanks to works [4, 16] and are related to Horn's hypergeometric function H_4 (see [23])

$$H_4(a,b;c,d;\mathbf{z})=\sum_{r,s=0}^{\infty}\frac{(a)_{2r+s}(b)_s}{(c)_r(d)_s}\frac{z_1^r}{r!}\frac{z_2^s}{s!},\quad |z_1|<p,\ |z_2|<q,$$

where $a,b,c,d\in\mathbb{C}$; $c,d\notin\{0,-1,-2,\ldots\}$; p and q are positive numbers such that $4p=(q-1)^2$ and $q\neq 1$; $(\alpha)_0=1$ and $(\alpha)_n=\alpha(\alpha+1)\ldots(\alpha+n-1)$, $\mathbf{z}=(z_1,z_2)\in\mathbb{C}^2$. The paper [4] provides the formal expansion

$$\frac{H_4(a,b;c,b;\mathbf{z})}{H_4(a+1,b;c+1,b;\mathbf{z})}=1-z_2-\cfrac{h_1z_1}{1-z_2-\cfrac{h_2z_1}{1-z_2-\cfrac{h_3z_1}{1-\ddots}}}, \tag{2}$$

where

$$h_k=\frac{(2c-a+k-1)(a+k)}{(c+k-1)(c+k)},\quad k\geq 1, \tag{3}$$

as well as,

$$\frac{H_4(a,d+1;c,d;\mathbf{z})}{H_4(a+1,d+1;c,d+1;\mathbf{z})}=1-\frac{d-a}{d}z_2-\cfrac{m_1z_1}{1-z_2-\cfrac{m_2z_1}{1-z_2-\cfrac{m_3z_1}{1-\ddots}}}, \tag{4}$$

where $m_1=\dfrac{2(a+1)}{c}$, $m_k=\dfrac{(2c-a+k-3)(a+k)}{(c+k-2)(c+k-1)}$ $(k\geq 2)$, and

$$\frac{H_4(a,d+1;c,d;\mathbf{z})}{H_4(a,d+2;c,d+1;\mathbf{z})}=1+\cfrac{v_0z_2}{1+v_1z_2+\cfrac{u_1z_1}{1+v_2z_2+\cfrac{u_2z_1}{1+\ddots}}}, \tag{5}$$

where $v_0 = \dfrac{a}{d(d+1)}$, $v_1 = \dfrac{a}{d+1} - 1$, $u_1 = -\dfrac{2(a+1)}{c}$,
$$v_k = -1, \quad u_k = -\frac{(2c-a+k-3)(a+k)}{(c+k-2)(c+k-1)} \quad (k \geq 2).$$

Some questions of convergence of the expansions (2), (4), and (5) were discussed in [4, 13, 14, 15]. Numerical aspects related to the backward recurrence algorithm for computing the approximants of continued fractions were considered in [7, 9, 27, 28, 30]. Some analogous results concerning branched continued fractions can be found in [19, 21, 22, 25, 31, 32].

2. Estimates of relative rounding error. In this section, we will establish an estimate of the relative rounding error produced by the backward recurrence algorithm in calculating the nth approximant of (2).

Let us recall the necessary concepts. Let n be an arbitrary fixed natural number. For each $1 \leq k \leq n$, let $\widehat{a}_{1,k-1}$ and $\widehat{a}_{0,k}$ denote rounded values of the elements $a_{1,k-1}$ and $a_{0,k}$, respectively, of a given branched continued fraction (1). The number

$$\widehat{f}_n = 1 + \widehat{a}_{1,0} + \cfrac{\widehat{a}_{0,1}}{1 + \widehat{a}_{1,1} + \cfrac{\widehat{a}_{0,2}}{1 + \widehat{a}_{1,2} + \cfrac{\widehat{a}_{0,3}}{1 + \ddots + \widehat{a}_{1,n-2} + \cfrac{\widehat{a}_{0,n-1}}{1 + \widehat{a}_{1,n-1} + \widehat{a}_{0,n}}}}}$$

is the computed (approximate) value of $f_n = G_0^{(n)}$.

Definition 1. A *numerical stability set* Ω is a set to which for any $\varepsilon > 0$ one can find $\delta > 0$ depending only on ε and Ω such that, for all $n \geq 1$
$$\left|\widehat{f}_n - f_n\right| < \varepsilon \cdot |f_n|$$
for every branched continued fraction (1) with all $a_{1,k-1}, a_{0,k} \in \Omega$ and $\widehat{a}_{1,k-1}, \widehat{a}_{0,k} \in \Omega$ such that, for all $k \geq 1$
$$\left|\frac{\widehat{a}_{1,k-1} - a_{1,k-1}}{a_{1,k-1}}\right| < \delta \quad \text{and} \quad \left|\frac{\widehat{a}_{0,k} - a_{0,k}}{a_{0,k}}\right| < \delta.$$

We set

$$G_n^{(n)}(\mathbf{z}) = 1, \quad n \geq 1, \tag{6}$$

and

$$G_k^{(n)}(\mathbf{z}) = 1 - z_2 - \cfrac{h_{k+1}z_1}{1 - z_2 - \cfrac{h_{k+2}z_1}{1 - \ddots - z_2 - \cfrac{h_{n-1}z_1}{1 - z_2 - h_n z_1}}}$$

for $1 \leq k \leq n-1$, $n \geq 2$, where h_k, $1 \leq k \leq n-1$, $n \geq 2$, are defined by (3). It follows that

$$G_k^{(n)}(\mathbf{z}) = 1 - z_2 - \frac{h_{k+1}z_1}{G_{k+1}^{(n)}(\mathbf{z})}, \quad 1 \leq k \leq n-1, \quad n \geq 2, \tag{7}$$

and nth approximant of (2) we write as

$$f_n(\mathbf{z})=1-z_2-\frac{h_1z_1}{G_1^{(k)}(\mathbf{z})}.$$

Let α_1, α_2, and β_k, $1\le k\le n$, denote the relative errors in the rounded values $\widehat{z}_1$, $\widehat{z}_2$, and $\widehat{h}_k$, $1\le k\le n$, of z_1, z_2, and h_k, $1\le k\le n$, respectively, so that

$$\widehat{z}_1=z_1(1+\alpha_1),\quad \widehat{z}_2=z_2(1+\alpha_2),\quad \widehat{h}_k=h_k(1+\beta_k),\quad 1\le k\le n. \tag{8}$$

Similarly, let $\varepsilon_k^{(n)}$, $0\le k\le n$, denote the relative errors in $\widehat{G}_k^{(n)}(\widehat{\mathbf{z}})$, the approximation to $G_k^{(n)}(\mathbf{z})$ from (6)–(7) and

$$G_0^{(n)}(\mathbf{z})=1-z_2-\frac{h_1z_1}{G_1^{(n)}(\mathbf{z})}$$

using $\widehat{z}_1$, $\widehat{z}_2$, and $\widehat{h}_k$, $1\le k\le n$. Thus,

$$\widehat{G}_k^{(n)}(\widehat{\mathbf{z}})=G_k^{(n)}(\mathbf{z})(1+\varepsilon_k^{(n)}),\quad 0\le k\le n, \tag{9}$$

and

$$\widehat{G}_n^{(n)}(\widehat{\mathbf{z}})=G_n^{(n)}(\mathbf{z})=1,\quad \text{and}\quad \varepsilon_n^{(n)}=0. \tag{10}$$

Also, for convenience, let $\widehat{\alpha}_1$, $\widehat{\alpha}_2$, $\widehat{\beta}_k$, $1\le k\le n$, and $\widehat{\varepsilon}_k^{(n)}$, $0\le k\le n$, denote the relative errors defined by $z_1=\widehat{z}_1(1+\widehat{\alpha}_1)$, $z_2=\widehat{z}_2(1+\widehat{\alpha}_2)$, $h_k=\widehat{h}_k(1+\widehat{\beta}_k)$, $1\le k\le n$, and

$$G_k^{(n)}(\mathbf{z})=\widehat{G}_k^{(n)}(\widehat{\mathbf{z}})(1+\widehat{\varepsilon}_k^{(n)}),\quad 0\le k\le n,$$

respectively.

Next, we establish recurrence relations for relative errors $\varepsilon_k^{(n)}$, $0\le k\le n-1$. For arbitrary k, $0\le k\le n-1$, one obtains

$$\begin{aligned}
\varepsilon_k^{(n)}&=\frac{\widehat{G}_k^{(n)}(\widehat{\mathbf{z}})-G_k^{(n)}(\mathbf{z})}{G_k^{(n)}(\mathbf{z})}=\frac{1}{G_k^{(n)}(\mathbf{z})}\left(1-\widehat{z}_2-\frac{\widehat{h}_{k+1}\widehat{z}_1}{\widehat{G}_{k+1}^{(n)}(\widehat{\mathbf{z}})}\right)-1=\\
&=\frac{1}{G_k^{(n)}(\mathbf{z})}\left(1-z_2(1+\alpha_2)-\frac{h_{k+1}(1+\beta_{k+1})z_1(1+\alpha_1)}{G_{k+1}^{(n)}(\mathbf{z})(1+\varepsilon_{k+1}^{(n)})}\right)-1=\\
&=\frac{1}{G_k^{(n)}(\mathbf{z})}-\frac{z_2(1+\alpha_2)}{G_k^{(n)}(\mathbf{z})}-\frac{h_{k+1}(1+\beta_{k+1})z_1(1+\alpha_1)(1+\widehat{\varepsilon}_{k+1}^{(n)})}{G_k^{(n)}(\mathbf{z})G_{k+1}^{(n)}(\mathbf{z})}-1.
\end{aligned}$$

It follows from (7) that

$$\frac{1}{G_k^{(n)}(\mathbf{z})}=1+\frac{z_2}{G_k^{(n)}(\mathbf{z})}+\frac{h_{k+1}z_1}{G_k^{(n)}(\mathbf{z})G_{k+1}^{(n)}(\mathbf{z})}.$$

Then,

$$\begin{aligned}
\varepsilon_k^{(n)}&=\frac{z_2}{G_k^{(n)}(\mathbf{z})}-\frac{z_2(1+\alpha_2)}{G_k^{(n)}(\mathbf{z})}-\frac{h_{k+1}z_1}{G_k^{(n)}(\mathbf{z})G_{k+1}^{(n)}(\mathbf{z})}((1+\beta_{k+1})(1+\alpha_1)(1+\widehat{\varepsilon}_{k+1}^{(n)})-1)=\\
&=-\frac{z_2\alpha_2}{G_k^{(n)}(\mathbf{z})}-\frac{h_{k+1}z_1}{G_k^{(n)}(\mathbf{z})\widehat{G}_{k+1}^{(n)}(\widehat{\mathbf{z}})}(\beta_{k+1}+\alpha_1+\beta_{k+1}\alpha_1)-\frac{h_{k+1}z_1}{G_k^{(n)}(\mathbf{z})G_{k+1}^{(n)}(\mathbf{z})}\widehat{\varepsilon}_{k+1}^{(n)}.
\end{aligned}$$

Thus, for each $0\le k\le n-1$,

$$\varepsilon_k^{(n)}=-\frac{z_2\alpha_2}{G_k^{(n)}(\mathbf{z})}-\frac{h_{k+1}z_1}{G_k^{(n)}(\mathbf{z})\widehat{G}_{k+1}^{(n)}(\widehat{\mathbf{z}})}(\beta_{k+1}+\alpha_1+\beta_{k+1}\alpha_1)-\frac{h_{k+1}z_1}{G_k^{(n)}(\mathbf{z})G_{k+1}^{(n)}(\mathbf{z})}\widehat{\varepsilon}_{k+1}^{(n)}. \tag{11}$$

Similarly, for relative errors $\widehat{\varepsilon}_k^{(n)}$, $0 \le k \le n-1$, one obtains

$$\widehat{\varepsilon}_k^{(n)} = -\frac{\widehat{z}_2\widehat{\alpha}_2}{\widehat{G}_k^{(n)}(\mathbf{z})} - \frac{\widehat{h}_{k+1}\widehat{z}_1}{\widehat{G}_k^{(n)}(\widehat{\mathbf{z}})G_{k+1}^{(n)}(\mathbf{z})}(\widehat{\beta}_{k+1} + \widehat{\alpha}_1 + \widehat{\beta}_{k+1}\widehat{\alpha}_1) - \frac{\widehat{h}_{k+1}\widehat{z}_1}{\widehat{G}_k^{(n)}(\widehat{\mathbf{z}})\widehat{G}_{k+1}^{(n)}(\widehat{\mathbf{z}})}\varepsilon_{k+1}^{(n)}. \tag{12}$$

Combining (11) and (12) with

$$g_k^{(n)}(\mathbf{z}) = -\frac{h_k z_1}{G_{k-1}^{(n)}(\mathbf{z})G_k^{(n)}(\mathbf{z})}, \quad \widehat{g}_k^{(n)}(\widehat{\mathbf{z}}) = -\frac{\widehat{h}_k\widehat{z}_1}{\widehat{G}_{k-1}^{(n)}(\widehat{\mathbf{z}})\widehat{G}_k^{(n)}(\widehat{\mathbf{z}})}, \quad 1 \le k \le n-1, \tag{13}$$

one easily obtains the relation

$$\varepsilon_0^{(n)} = \sum_{k-1}^{n} \frac{(-1)^k}{\widetilde{G}_{k-1}^{(n)}}\left(z_{2,k}\alpha_{2,k} + \frac{\widetilde{h}_k z_{1,k}(\widetilde{\beta}_k + \alpha_{1,k} + \widetilde{\beta}_k\alpha_{1,k})}{\widetilde{G}_k^{(n)}}\right)\prod_{r=1}^{k-1}\widetilde{g}_r^{(n)}, \tag{14}$$

where, for $p = 1, 2$, $\quad z_{p,k} = \begin{cases} \widehat{z}_p, & \text{if } k \text{ even}, \\ z_p, & \text{if } k \text{ odd}, \end{cases} \qquad \alpha_{p,k} = \begin{cases} \widehat{\alpha}_p, & \text{if } k \text{ even}, \\ \alpha_p, & \text{if } k \text{ odd}, \end{cases}$ and

$$\widetilde{h}_k = \begin{cases} \widehat{h}_k, & \text{if } k \text{ even}, \\ h_k, & \text{if } k \text{ odd}, \end{cases} \qquad \widetilde{g}_k^{(n)} = \begin{cases} g_k^{(n)}(\mathbf{z}), & \text{if } k \text{ even}, \\ \widehat{g}_k^{(n)}(\widehat{\mathbf{z}}), & \text{if } k \text{ odd}, \end{cases} \qquad \widetilde{G}_k^{(n)} = \begin{cases} G_k^{(n)}(\mathbf{z}), & \text{if } k \text{ even}, \\ \widehat{G}_k^{(n)}(\widehat{\mathbf{z}}), & \text{if } k \text{ odd}. \end{cases}$$

For convenience, we set $\varepsilon_n = \varepsilon_0^{(n)}$. Now we are ready to prove our main result.

Theorem 1. *Let there exist a constant* α, $0 < \alpha < 1$, *such that*

$$|\alpha_1| \le \alpha, \quad |\alpha_2| \le \alpha, \quad \textit{and} \quad |\beta_k| \le \alpha \quad \textit{for all} \quad k \ge 1, \tag{15}$$

where α_1, α_2, *and* β_k, $k \ge 1$, *are relative errors of* z_1, z_2, *and* h_k, $k \ge 1$, *respectively, which are defined in* (8) *for all* $n \ge 1$. *Then:*
(A) The set

$$\mathrm{H}_{h,l} = \left\{\mathbf{z} \in \mathbb{C}^2 : \ |z_1| < l(1-l)/(2h), \ |z_2| < (1-l)/2\right\}, \tag{16}$$

where

$$h = \max_{k\in\mathbb{N}}\{|h_k|, \ |\widehat{h}_k|\}, \quad l \in (0, 1/3) \cup (1/3, 1), \tag{17}$$

forms the numerical stability set of the branched continued fraction (2).
(B) If ε_n *denotes the relative errors of* n*th approximant of* (2), *then, for* $n \ge 1$,

$$|\varepsilon_n| \le \frac{4\alpha}{(1 + l + |3l-1|)(1-\alpha)}\left(\frac{1-l}{2} + \frac{2l(1-l)}{1+l+|3l-1|}\left(2 + \frac{\alpha}{1-\alpha}\right)\right)\frac{1-\eta^n}{1-\eta}, \tag{18}$$

where

$$\eta = \begin{cases} 2l/(1-l), & \textit{if } \ 1 < l < 1/3, \\ (1-l)/(2l), & \textit{if } \ 1/3 < l < 1. \end{cases}$$

Remark 1. It follows from [15, Theorem 2] that if a and c are complex constants such that

$$|h_k| + \mathrm{Re}(h_k) \le pq(1-q) \quad \text{for all} \quad k \ge 1,$$

where h_k, $k \ge 1$, are defined by (3), $c \notin \{0, -1, -2, \ldots\}$, p is a positive number, $0 < q < 1$, then the branched continued fraction (2) converges uniformly on every compact subset of the domain (16).

Proof of Theorem 1. We consider the periodic continued fraction

$$(1+l)/2 - \cfrac{l(1-l)/2}{(1+l)/2 - \cfrac{l(1-l)/2}{(1+l)/2 - \cfrac{l(1-l)/2}{(1+l)/2 - \ddots}}}, \tag{19}$$

which is equivalent to

$$(1+l)/2 - \cfrac{l(1-l)/(1+l)}{1 - \cfrac{2l(1-l)/(1+l)^2}{1 - \cfrac{2l(1-l)/(1+l)^2}{1 - \ddots}}}, \tag{20}$$

since $(1+l)/2 \ne 0$ (see, [26, Section 2.3]).

It is easy to show that $-\frac{1}{4} < -\frac{2l(1-l)}{(1+l)^2}$, i.e. the elements of (20) satisfy Theorem 3.2 in [26]. According to this theorem, the continued fraction (20) converges, and its value is

$$f^* = \frac{1+l}{2}\left(1 - \frac{1 - 2\sqrt{1/4 - 2l(1-l)/(1+l)^2}}{2}\right) =$$
$$= \frac{1+l}{2}\left(1 - \frac{1+l-|1-3l|}{2(1+l)}\right) = \frac{1+l+|1-3l|}{4}.$$

The continued fraction (19), as equivalent to (20), also converges to the value f^*. In addition, it is easy to show that the approximants

$$f_n^* = (1+l)/2 - \cfrac{l(1-l)/2}{(1+l)/2 - \cfrac{l(1-l)/2}{(1+l)/2 - \ddots - \cfrac{l(1-l)/2}{(1+l)/2}}}, \quad n \ge 1,$$

forms a monotonically descending sequence.

Let n be an arbitrary fixed natural number. Let us prove that

$$|G_k^{(n)}(\mathbf{z})| > f_{n-k}^*, \quad 0 \le k \le n-1,$$

where $G_k^{(n)}(\mathbf{z})$, $0 \le k \le n-1$, are defined by (6)–(7).

If $k = n-1$, we have

$$|G_{n-1}^{(n)}(\mathbf{z})| \ge 1 - |z_2| - |h_n||z_1| > \frac{1+l}{2} - \frac{l(1-l)}{2} > (1+l)/2 - \frac{l(1-l)/2}{(1+l)/2} = f_1^*.$$

Assuming that the inequality (20) is true if $k = s+1 \le n-1$. Then, for $k = s$ from (7) we obtain

$$|G_s^{(n)}(\mathbf{z})| \geq 1 - |z_2| - \frac{|h_{s+1}||z_1|}{|G_{s+1}^{(n)}(\mathbf{z})|} > (1+l)/2 - \frac{l(1-l)/2}{f^*_{n-s-1}} = f^*_{n-s}.$$

Since $f_n^* > f^*$ for all $n \geq 1$, then $|G_k^{(n)}(\mathbf{z})| > f^*$ for each $0 \leq k \leq n-1$. Considering $z_1 \neq 0$, let us estimate the values $g_k^{(n)}(\mathbf{z})$, $1 \leq k \leq n-1$, which are defined in (13).

For any k, $1 \leq k \leq n-1$, one obtains

$$|g_k^{(n)}(\mathbf{z})| = \left|\frac{h_k z_1}{G_{k+1}^{(n)}(\mathbf{z})G_k^{(n)}(\mathbf{z})}\right| = \left|\frac{h_k z_1/G_k^{(n)}(\mathbf{z})}{1 - z_2 - h_k z_1/G_k^{(n)}(\mathbf{z})}\right| = \frac{1}{\left|\dfrac{1-z_2}{h_k z_1}G_k^{(n)}(\mathbf{z}) - 1\right|} \leq$$

$$\leq \frac{1}{\dfrac{1-|z_2|}{|h_k||z_1|}|G_k^{(n)}(\mathbf{z})| - 1} < \frac{1}{\dfrac{(1+l)(1+l+|1-3l|)}{4l(l-1)} - 1} = \begin{cases} \dfrac{2l}{1-l}, & \text{if } 1 < l < \dfrac{1}{3}, \\ \dfrac{1-l}{2l}, & \text{if } \dfrac{1}{3} < l < 1. \end{cases}$$

Now since $\widehat{\mathbf{z}} = (\widehat{z}_1, \widehat{z}_2) \in \Omega^{h,l}$, we have

$$|\widehat{G}_k^{(n)}(\widehat{\mathbf{z}})| > f^*_{n-k} \quad \text{for each} \quad 0 \leq k \leq n-1, \tag{21}$$

and

$$\widehat{g}_k^{(n)}(\widehat{\mathbf{z}}) < \eta \quad \text{for each} \quad 1 \leq k \leq n-1, \tag{22}$$

where $\widehat{G}_k^{(n)}(\widehat{\mathbf{z}})$, $0 \leq k \leq n-1$, and $\widehat{g}_k^{(n)}(\widehat{\mathbf{z}})$, $1 \leq k \leq n-1$, are defined in (9)–(10) and (13), respectively. Further, from the conditions of this theorem it follows

$$|z_{1,k}| < \frac{l(1-l)}{2h}, \quad |z_{2,k}| < \frac{1-l}{2}, \quad |\widetilde{h}_k| \leq h \quad \text{for each} \quad 1 \leq k \leq n.$$

Thus, from (14) we have

$$|\varepsilon_n| \leq \sum_{k-1}^{n} \frac{1}{|\widetilde{G}_{k-1}^{(n)}|}\left(|z_{2,k}||\alpha_{2,k}| + \frac{|\widetilde{h}_k||z_{1,k}|(|\widetilde{\beta}_k| + |\alpha_{1,k}| + |\widetilde{\beta}_k||\alpha_{1,k}|)}{|\widetilde{G}_k^{(n)}|}\right)\prod_{r=1}^{k-1}|\widetilde{g}_r^{(n)}|.$$

Using (21)–(22), we get

$$|\varepsilon_n| \leq \frac{4\alpha}{(1+l+|3l-1|)(1-\alpha)}\left(\frac{1-l}{2} + \frac{2l(1-l)}{1+l+|3l-1|}\left(2 + \frac{\alpha}{1-\alpha}\right)\right)\sum_{k=1}^{n}\eta^{k-1},$$

which is equal to (18), since $\sum_{k=1}^n \eta^{k-1} = \frac{1-\eta^n}{1-\eta}$.

Finally, it follows from (18) that there exists a constant C such that $|\varepsilon_n| \leq \alpha C/(1-\alpha)$ for all $n \geq 1$. It is easy to show that, if

$$|\alpha_1| \leq \alpha < \frac{\varepsilon}{\varepsilon + C}, \quad |\alpha_2| \leq \alpha < \frac{\varepsilon}{\varepsilon + C},$$

and $|\beta_k| \leq \alpha < \varepsilon/(\varepsilon + C)$ for all $k \geq 1$, where ε is an arbitrary positive constant, then $|\varepsilon_n| < \varepsilon$ for all $n \geq 1$. This fact proves that the conditions from Definition 1 are fulfilled. □

From Theorem 1 we have the following.

Corollary 1. *Let there exist a constant α, $0 < \alpha < 1$, satisfying* (15), *where α_1, α_2, and β_k, $k \geq 1$, are relative errors of z_1, z_2, and h_k, $k \geq 1$, respectively, of the branched continued fraction*

$$\cfrac{1}{1-z_2-\cfrac{h_1z_1}{1-z_2-\cfrac{h_2z_1}{1-\ddots}}}, \tag{23}$$

where $h_1 = 2/c$, $h_k = \dfrac{k(2c+k-3)}{(c+k-2)(c+k-1)}$ for all $k \geq 2$. Then:

(A) The set (16), *where h and l are defined in* (17), *forms the numerical stability set of* (23).

(B) If ε_n denotes the relative errors of the nth approximant of the branched continued fraction (23), *then estimate* (18) *holds for all $n \geq 1$.*

Remark 2. Results similar to Theorem 1 can also be obtained for the other two expansions in (4) and (5). In the general case, the problem of studying the numerical stability set of all three expansions remains open.

3. Numerical experiments. To illustrate the numerical stability of the backward recurrence algorithm, we describe the numerical examples taken from the branched continued fraction representation of the function (see [4])

$$H_4(1,b;1,b;\mathbf{z}) = \sqrt{(1-z_2)^2-4z_1} = \cfrac{1}{1-z_2-\cfrac{2z_1}{1-z_2-\cfrac{z_1}{1-z_2-\cfrac{z_1}{1-z_2-\ddots}}}}. \tag{24}$$

We considered the values of the nth approximants $f_n(\mathbf{z})$, $1 \leq n \leq 100$, correctly rounded to 28 decimal digits for the points $(0.125, 0.25)$ and $(0.0625, -0.25)$, respectively. We also considered an approximation to each $f_n(\mathbf{z})$ obtained by the backward recurrence algorithm, correctly rounded to 14 decimal digits. We take the values $\widehat{f_n}(0.125, 0.25)$ correctly rounded to 14 decimal places for all $1 \leq n \leq 100$.

Further, calculations of $f_n(0.125, 0.25)$ and $\widehat{f_n}(0.125, 0.25)$, for $55 \leq n \leq 96$, showed that $|\widehat{f_n}(0.125, 0.25) - f_n(0.125, 0.25)|/|f_n(0.125, 0.25)|$ decreases and for $97 \leq n \leq 100$ are equal zero, as well as for $n = 1, 2$. The relative rounding error at the point $(0.0625, -0.25)$ does not decrease or increase for $1 \leq n \leq 100$. In the case of the backward recurrence algorithm, $\widehat{f_n}(0.0625, -0.25)$ remains correctly rounded to 14 decimal places for all $1 \leq n \leq 100$. Finally, from Corollary 1, one can obtain rigorous bounds for the relative rounding error $|\varepsilon_n|$ for each $n \geq 1$, which are entirely consistent with those found numerically in these examples.

Our calculations were performed using Maple software 2022.2 for Windows.

4. Conclusions. This paper concerns the establishment of the numerical stability sets of the branched continued fraction in the domains of its convergence. An estimate of the relative rounding error, produced by the backward recurrence algorithm in calculating an nth approximant of the branched continued fraction expansion of Horn's hypergeometric function H_4, is established. It has provided to investigate the numerical stability of the bi-disc.

The considered numerical experiments are entirely consistent with theoretical calculations. Both of them, in particular, show that the stability of the backward recurrence algorithm depends not only on the calculation value of the elements of the branched continued fraction but also on the domain of convergence. Further research is to study wider sets of numerical stability.

REFERENCES

1. T. Antonova, R. Dmytryshyn, V. Goran, *On the analytic continuation of Lauricella-Saran hypergeometric function* $F_K(a_1, a_2, b_1, b_2; a_1, b_2, c_3; \mathbf{z})$, Mathematics, **11** (2023), 4487. http://dx.doi.org/10.3390/math11214487
2. T. Antonova, R. Dmytryshyn, V. Kravtsiv, *Branched continued fraction expansions of Horn's hypergeometric function* H_3 *ratios*, Mathematics, **9** (2021), 148. http://dx.doi.org/10.3390/math9020148
3. T. Antonova, R. Dmytryshyn, S. Sharyn, *Generalized hypergeometric function* ${}_3F_2$ *ratios and branched continued fraction expansions*, Axioms, **10** (2021), 310. http://dx.doi.org/10.3390/axioms10040310
4. T. Antonova, R. Dmytryshyn, I.-A. Lutsiv, S. Sharyn, *On some branched continued fraction expansions for Horn's hypergeometric function* $H_4(a, b; c, d; z_1, z_2)$ *ratios*, Axioms, **12** (2023), 299. http://dx.doi.org/10.3390/axioms12030299
5. T. Antonova, R. Dmytryshyn, S. Sharyn, *Branched continued fraction representations of ratios of Horn's confluent function* H_6, Constr. Math. Anal., **6** (2023), 22–37. http://dx.doi.org/10.33205/cma.1243021
6. T.M. Antonova, *On convergence of branched continued fraction expansions of Horn's hypergeometric function* H_3 *ratios*, Carpathian Math. Publ., **13**, (2021), 642–650. https://doi.org/10.15330/cmp.13.3.642-650
7. G. Blanch, *Numerical evaluation of continued fractions*, SIAM Review, **6** (1964), 383–421. http://dx.doi.org/10.1137/1006092
8. D.I. Bodnar, Branched Continued Fractions, Naukova Dumka, Kyiv, 1986. (in Russian)
9. A. Cuyt, P. Van der Cruyssen, *Rounding error analysis for forward continued fraction algorithms*, Comput. Math. Appl., **11** (1985), 541–564. http://dx.doi.org/10.1016/0898-1221(85)90037-9
10. D.I. Bodnar, R.I. Dmytryshyn, *Multidimensional associated fractions with independent variables and multiple power series*, Ukr. Math. Zhurn., **71** (2019), 325–339. (in Ukrainian); Engl. transl.: Ukrainian Math. J., **71** (2019), 370–386. http://dx.doi.org/10.1007/s11253-019-01652-5
11. D.I. Bodnar, O.S. Manzii, *Expansion of the ratio of Appel hypergeometric functions* F_3 *into a branching continued fraction and its limit behavior*, Mat. method. and fiz.-mech. polya, **41** (1998), 12–16. (in Ukrainian); Engl. transl.: J. Math. Sci., **107** (2001), 3550–3554. http://dx.doi.org/10.1023/A:1011977720316
12. R. Dmytryshyn, V. Goran, *On the analytic extension of Lauricella–Saran's hypergeometric function* F_K *to symmetric domains*, Symmetry, **16** (2024), 220. http://dx.doi.org/10.3390/sym16020220
13. R. Dmytryshyn, I.-A. Lutsiv, O. Bodnar, *On the domains of convergence of the branched continued fraction expansion of ratio* $H_4(a, d + 1; c, d; \mathbf{z})/H_4(a, d + 2; c, d + 1; \mathbf{z})$, Res. Math., **31** (2023), 19–26. http://dx.doi.org/10.15421/242311
14. R. Dmytryshyn, I.-A. Lutsiv, M. Dmytryshyn, C. Cesarano, *On some domains of convergence of branched continued fraction expansions of ratios of Horn hypergeometric functions* H_4, Ukr. Math. Zhurn., 2023, (accepted). (in Ukrainian)
15. R. Dmytryshyn, I.-A. Lutsiv, M. Dmytryshyn, *On the analytic extension of the Horn's hypergeometric function* H_4, Carpathian Math. Publ., **16** (2024), 32–39. http://dx.doi.org/10.15330/cmp.16.1.32-39
16. R.I. Dmytryshyn, I.-A.V. Lutsiv *Three- and four-term recurrence relations for Horn's hypergeometric function* H_4, Res. Math., **30** (2022), 21–29. http://dx.doi.org/10.15421/242203
17. R.I. Dmytryshyn, S.V. Sharyn, *Approximation of functions of several variables by multidimensional S-fractions with independent variables*, Carpathian Math. Publ., **13** (2021), 592–607. http://dx.doi.org/10.15330/cmp.13.3.592-607
18. R.I. Dmytryshyn, *Two-dimensional generalization of the Rutishauser qd-algorithm*, Mat. method. and fiz.-mech. polya, **56** (2013), 6–11. (in Ukrainian); Engl. transl.: J. Math. Sci., **208** (2015), 301–309. http://dx.doi.org/10.1007/s10958-015-2447-9

19. V.R. Hladun, D.I. Bodnar, R.S. Rusyn, *Convergence sets and relative stability to perturbations of a branched continued fraction with positive elements*, Carpathian Math. Publ., **16** (2024), 16–31. http://dx.doi.org/10.15330/cmp.16.1.16-31
20. V.R. Hladun, N.P. Hoyenko, O.S. Manzij, L. Ventyk, *On convergence of function $F_4(1,2;2,2;z_1,z_2)$ expansion into a branched continued fraction*, Math. Model. Comput., **9** (2022), 767–778. http://dx.doi.org/10.23939/mmc2022.03.767
21. V.R. Hladun, Stability analysis to perturbations of branched continued fractions, PhD Thesis on Mathematical Analysis, Ivan Franko Lviv National University, Lviv, 2007. (in Ukrainian).
22. V.R. Hladun, *Some sets of relative stability under perturbations of branched continued fractions with complex elements and a variable number of branches*, Mat. method. and fiz.-mech. polya, **57** (2014), 14–24. (in Ukrainian); Engl. transl.: J. Math. Sci., **215** (2016), 11–25. http://dx.doi.org/10.1007/s10958-016-2818-x
23. J. Horn, *Hypergeometrische funktionen zweier veränderlichen*, Math. Ann., **105** (1931), 381–407. http://dx.doi.org/10.1007/BF01455825
24. N. Hoyenko, T. Antonova, S. Rakintsev, *Approximation for ratios of Lauricella–Saran fuctions F_S with real parameters by a branched continued fractions*, Math. Bul. Shevchenko Sci. Soc., **8** (2011), 28–42. (in Ukrainian).
25. N.P. Hoyenko, V.R. Hladun, O.S. Manzij, *On the infinite remains of the Nörlund branched continued fraction for Appell hypergeometric functions*, Carpathian Math. Publ., **6** (2014), 11–25. (in Ukrainian) http://dx.doi.org/10.15330/cmp.6.1.11-25
26. W.B. Jones, W.J. Thron, Continued Fractions: Analytic Theory and Applications, Addison-Wesley Pub. Co., Reading, 1980.
27. W.B. Jones, W.J. Thron, *Numerical stability in evaluating continued fractions*, Math. Comp., **28** (1974), 795–810. http://dx.doi.org/10.2307/2005701
28. W.B. Jones, W.J. Thron, *Rounding error in evaluating continued fractions*, Proceedings of the ACM, San Diego, (1974), 11–19.
29. H. Lima, *Multiple orthogonal polynomials associated with branched continued fractions for ratios of hypergeometric series*, Adv. Appl. Math., **147** (2023), 102505. http://dx.doi.org/10.1016/j.aam.2023.102505
30. N. Macon, M. Baskervill, *On the generation of errors in the digital evaluation of continued fractions*, J. Assoc. Comput. Math., **3** (1956), 199–202. http://dx.doi.org/10.1145/320831.320838
31. O. Manziy, V. Hladun, L. Ventyk, *The algorithms of constructing the continued fractions for any rations of the hypergeometric Gaussian functions*, Math. Model. Comput., **4** (2017), 48–58. http://dx.doi.org/10.23939/mmc2017.01.048
32. M.O. Nedashkovskyi, *On the convergence and computational stability of branched continued fractions of certain types*, Mat. Metody Fiz. Mekh. Polya, **20** (1984), 27–31. (in Russian)
33. M. Petreolle, A.D. Sokal, *Lattice paths and branched continued fractions II. Multivariate Lah polynomials and Lah symmetric functions*, Eur. J. Combin., **92** (2021), 103235. http://dx.doi.org/10.1016/j.ejc.2020.103235

Vasyl Stefanyk Precarpathian National University
Ivano-Frankivsk, Ukraine
dmytryshynr@hotmail.com

International Telematic University UNINETTUNO
Roma, Italy
c.cesarano@uninettunouniversity.net

Vasyl Stefanyk Precarpathian National University
Ivano-Frankivsk, Ukraine
lutsiv.ilona@gmail.com

West Ukrainian National University
Ternopil, Ukraine
m.dmytryshyn@wunu.edu.ua

Received 21.12.2023
Revised 27.02.2024