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ON THE h-MEASURE OF AN EXCEPTIONAL SET IN FENTON-TYPE THEOREM FOR TAYLOR-DIRICHLET SERIES

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We consider the class $S(\lambda,\beta,\tau)$ of convergent for all $x\geq 0$ Taylor-Dirichlet type series of the form

$$F(x)=\sum_{n=0}^{+\infty}b_ne^{x\lambda_n+\tau(x)\beta_n},\ b_n\geq 0\ (n\geq 0),$$

where $\tau\colon [0,+\infty)\to(0,+\infty)$ is a continuously differentiable non-decreasing function, $\lambda=(\lambda_n)$ and $\beta=(\beta_n)$ are such that $\lambda_n\geq 0,\beta_n\geq 0$ $(n\geq 0)$. In the paper we give a partial answer to a question formulated by Salo T.M., Skaskiv O.B., Trusevych O.M. on International conference "Complex Analysis and Related Topics" (Lviv, September 23-28, 2013) ([2]). We prove the following statement: For each increasing function $h(x)\colon [0,+\infty)\to(0,+\infty)$, $h'(x)\nearrow+\infty$ $(x\to+\infty)$, every sequence $\lambda=(\lambda_n)$ such that

$$\sum_{n=0}^{+\infty}\frac{1}{\lambda_{n+1}-\lambda_n}<+\infty$$

and for any non-decreasing sequence $\beta=(\beta_n)$ such that $\beta_{n+1}-\beta_n\leq\lambda_{n+1}-\lambda_n$ $(n\geq 0)$ there exist a function $\tau(x)$ such that $\tau'(x)\geq 1$ $(x\geq x_0)$, a function $F\in S(\alpha,\beta,\tau)$, a set E and a constant $d>0$ such that h-meas $E:=\int_E dh(x)=+\infty$ and

$$(\forall x\in E)\colon\ F(x)>(1+d)\mu(x,F),$$

where $\mu(x,F)=\max\{|a_n|e^{x\lambda_n+\tau(x)\beta_n}\colon n\geq 0\}$ is the maximal term of the series.

At the same time, we also pose some open questions and formulate one conjecture.

1. Main result. In this article, we give a partial answer to a question formulated on International conference "Complex Analysis and Related Topics" (Lviv, September 23-28, 2013) ([2]). Let $\tau\colon [0,+\infty)\to(0,+\infty)$ be continuously differentiable non-decreasing function, $\lambda=(\lambda_n)$ and $\beta=(\beta_n)$ be such that $\lambda_n\geq 0,\beta_n\geq 0$ $(n\geq 0)$, and $S(\lambda,\beta,\tau)$ be the class of convergent for all $x\geq 0$ Taylor-Dirichlet type series of the form

$$F(x)=\sum_{n=0}^{+\infty}b_ne^{x\lambda_n+\tau(x)\beta_n},\quad b_n\geq 0\ (n\geq 0). \tag{1}$$

For $F\in S(\lambda,\beta,\tau)$ and $x\geq 0$ we denote by $\mu(x,F)=\max\{|a_n|e^{x\lambda_n+\tau(x)\beta_n}\colon n\geq 0\}$ the maximal term of the series, and by $\nu(x,F)=\max\{n\in\mathbb{N}_0\colon |a_n|e^{x\lambda_n+\tau(x)\beta_n}=\mu(x,F)\}$ the central index in the case when the max exists.

A theorem from paper [1] implies the following statement.

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Theorem 1 (Velychko, Skaskiv, 1989). *Let $\lambda=(\lambda_n), \beta=(\beta_n)$ be increasing sequences. If*

$$\sum_{n=0}^{+\infty}\frac{1}{\lambda_{n+1}-\lambda_n}<+\infty \tag{2}$$

and $F\in S(\lambda,\beta,\tau)$ then

$$F(x)=(1+o(1))\mu(x,F) \tag{3}$$

as $x\to+\infty$ outside some set $E\subset[0,+\infty)$ of finite Lebesgue measure, i.e. $\int_E dx<+\infty$.

In [2], it was posed the following conjecture.

Conjecture 1 ([2]). *For every sequences λ and β, functions τ, h, $\frac{h(x)}{x}\to+\infty$ $(x\to+\infty)$, there exist a function $F\in S(\lambda,\beta,\tau)$, a set E and a constant $d>0$ such that h-meas $E:=\int_E dh(x)=+\infty$ and $\forall x\in E$ the inequality $F(x)>(1+d)\mu(x,F)$ holds.*

In this note, we will prove the following statement.

Theorem 2. *For every increasing function $h(x)\colon[0,+\infty)\to(0,+\infty)$, $h'(x)\nearrow+\infty$ $(x\to+\infty)$, every sequence $\lambda=(\lambda_n)$ such that condition (2) holds and for every non-decreasing sequence $\beta=(\beta_n)$ such that $\beta_{n+1}-\beta_n\le\lambda_{n+1}-\lambda_n$ $(n\ge0)$ there exist a function $\tau(x)$ such that $\tau'(x)\ge1$, a function $F\in S(\alpha,\beta,\tau)$, a set E and $d>0$ such that h-meas $E=+\infty$ and*

$$(\forall x\in E):\quad F(x)>(1+d)\mu(x,F).$$

Proof. There exists a sequence (c_n) such that $c_n\uparrow+\infty$ and

$$\sum_{n=0}^{+\infty}\frac{c_n}{\lambda_{n+1}-\lambda_n}=+\infty. \tag{4}$$

We define the sequence $(\varkappa_n)$ in such a way that the following conditions are fulfilled:

$$\varkappa_0=0,\quad h'(\varkappa_n)\ge c_n,\quad \varkappa_n\ge\varkappa_{n-1}+\tfrac{2c}{\lambda_n-\lambda_{n-1}}\ (n\ge1),\quad c>0.$$

It is clear that the conditions of choice the sequence $(\varkappa_n)$ are not contradictory. In addition, since $c_n\uparrow+\infty$ $(n\to+\infty)$ and $h'(x)\to+\infty$ $(x\to+\infty)$, one has $\varkappa_n\uparrow+\infty$ $(n\to+\infty)$. Let us consider the function $\tau(x)$ such that $\tau'(x)=\frac{\lambda_{n+1}-\lambda_n}{\beta_{n+1}-\beta_n}$ for $x\in[\varkappa_n+\frac{c}{\lambda_{n+1}-\lambda_n},\varkappa_{n+1}]$ and $\tau'(x)=l_nx+k_n$ for $x\in[\varkappa_n,\varkappa_n+\frac{c}{\lambda_{n+1}-\lambda_n}]$ such that

$$l_n\varkappa_n+k_n=\frac{\lambda_n-\lambda_{n-1}}{\beta_n-\beta_{n-1}},\ l_n\Big(\varkappa_n+\frac{c}{\lambda_{n+1}-\lambda_n}\Big)+k_n=\frac{\lambda_{n+1}-\lambda_n}{\beta_{n+1}-\beta_n}.$$

It is easy to see that $\tau'(x)\ge1$ for all $x\ge\varkappa_1$.

We put

$$\ln\frac{b_n}{b_{n+1}}=\varkappa_{n+1}(\lambda_{n+1}-\lambda_n)+\tau(\varkappa_{n+1})(\beta_{n+1}-\beta_n).$$

Now it is easy to check that the function F of form (1) belongs to the class $S(\alpha,\beta,\tau)$.

Indeed, from the condition of the choice $\min\{\varkappa_n,\tau(\varkappa_n)\}\to+\infty$ $(n\to+\infty)$, therefore

$$\lim_{n\to+\infty}\frac{\ln b_n-\ln b_{n+1}}{\lambda_{n+1}-\lambda_n+\beta_{n+1}-\beta_n}=+\infty,$$

hence by Stolz-Cesáro theorem

$$\Delta_n := -\frac{\ln b_n}{\lambda_n+\beta_n} \to +\infty \quad (n\to+\infty).$$

From condition (2), using the Cauchy–Bunyakovsky–Schwarz's inequality, we obtain

$$m^2 = \left(\sum_{n=0}^{m-1}\frac{1}{\sqrt{\lambda_{n+1}-\lambda_n}}\sqrt{\lambda_{n+1}-\lambda_n}\right)^2 \le$$
$$\le \sum_{n=0}^{m-1}\frac{1}{\lambda_{n+1}-\lambda_n}\cdot\sum_{n=0}^{m-1}(\lambda_{n+1}-\lambda_n) \le K\lambda_m \quad (m\ge 1),\ K<+\infty.$$

In particular, $(\lambda_n+\beta_n)\ge\lambda_n\ge n$ for all sufficiently large $n\ge n_0$. So, for any $x>0$ we get

$$b_n+\exp\{x\lambda_n+\tau(x)\beta_n\}\le b_n+\exp\{(\lambda_n+\beta_n)\cdot\max\{x,\tau(x)\}\} =$$
$$=\exp\{-(\Delta_n-\max\{x,\tau(x)\})(\lambda_n+\beta_n)\}\le e^{-n}$$

as $n\to+\infty$. Hence, the function F of form (1) belongs to the class, i.e. $F\in S(\lambda,\beta,\tau)$.

Next, it is easy to prove that for any $x\in[\varkappa_n,\varkappa_{n+1})=I_n$ the central index $\nu(x,F)=n$. Therefore, for every $x\in I_n=[\varkappa_{n+1}-\frac{c}{\lambda_{n+1}-\lambda_n},\varkappa_{n+1})$ we obtain

$$\frac{F(x)}{\mu(x,F)}-1\ge\frac{b_{n+1}}{b_n}e^{x(\lambda_{n+1}-\lambda_n)+\tau(x)(\beta_{n+1}-\beta_n)} =$$
$$=\exp\Big((x-\varkappa_{n+1})(\lambda_{n+1}-\lambda_n)-(\tau(\varkappa_{n+1})-\tau(x))(\beta_{n+1}-\beta_n)\Big)\ge$$
$$\ge\exp\Big(-c-\frac{c}{\lambda_{n+1}-\lambda_n}\tau'(\theta_n)(\beta_{n+1}-\beta_n)\Big)=\exp(-2c)>0$$

because $\theta_n\in(\varkappa_{n+1}-\frac{c}{\lambda_{n+1}-\lambda_n},\varkappa_{n+1})$ and $\tau'(\theta_n)=\tau'(\varkappa_{n+1})=\frac{\lambda_{n+1}-\lambda_n}{\beta_{n+1}-\beta_n}$.

Now we consider h-meas of the set $E=\bigcup_{n=1}^{+\infty}I_n$. Since $h'(\varkappa_n)\ge c_n$ $(n\ge 0)$, by condition (4) we have

$$h\text{-meas } E=\int_E dh(x)=\sum_{n=1}^{+\infty}\int_{I_n}dh(x)=\sum_{n=1}^{+\infty}\left(h(\varkappa_{n+1})-h\left(\varkappa_{n+1}-\frac{c}{\lambda_{n+1}-\lambda_n}\right)\right)\ge$$
$$\ge\sum_{n=1}^{+\infty}\frac{c}{\lambda_{n+1}-\lambda_n}h'(\varkappa_n)\ge c\cdot\sum_{n=1}^{+\infty}\frac{c_n}{\lambda_{n+1}-\lambda_n}=+\infty.$$

□

Remark 1. The assertion of Theorem 2 for the class $S(\lambda):=S(\lambda,0,0)$, that is, for the entire Dirichlet series, it was proved earlier in paper [3].

2. Open problems and conjectures.

Given Theorem 2, the following questions arise.

Question 1. *Is the statement of conjecture 1 correct in its entirety?*

Question 2 ([2]). *Let $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function such that $\frac{h(x)}{x} \to +\infty$ $(x \to +\infty)$. What are necessary and sufficient conditions that relationship (3) holds for $x \to +\infty$ $(x \notin E, h$-meas $E < +\infty)$ for every function $F \in S(\alpha, \beta, \tau)$?*

In papers [5, 6] we find the following theorem.

Theorem 3 ([5, 6]). *Let h be a differentiable function with $h'(x) \uparrow +\infty$ $(x \to +\infty)$ and φ be the inverse function to the continuous positive function Φ which increases to $+\infty$ on $[0, +\infty)$. If*

$$(\forall b > 0): \quad \sum_{k=0}^{+\infty} \frac{1}{\lambda_{k+1} - \lambda_k} h'\Big(\varphi(\lambda_k) + \frac{b}{\lambda_{k+1} - \lambda_k}\Big) < +\infty, \tag{5}$$

then for all $F \in S(\Lambda)$ such that $\ln \mu(x, F) \geq x\Phi(x)$ asymptotic relation (3) holds as $x \to +\infty$ outside some set E of finite h-measure.

Conjecture 2. *A statement similar to Theorem 3 is also true in the class $S(\lambda, \beta, \tau)$.*

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REFERENCES

1. Velychko S.D., Skaskiv O.B. *Asymptotic properties of one class of functional series*, Visn. Lviv. Un-ty, Ser. Mekh.-Math., **32** (1989), 50–51. (in Ukrainian)
2. Salo T.M., Skaskiv O.B., Trusevych O.M. *About exceptional sets in Fenton's type theorem*, Complex Analysis and Related Topics: Abstracts, International conf. (Lviv, September 23-28, 2013), Lviv, 2013, 67–68.
3. Salo T.M., Skaskiv O.B., Trakalo O.M. *On the best possible description of exceptional set in Wiman-Valiron theory for entire function*, Mat. Stud. **16** (2001), №2, 131–140. http://matstud.org.ua/texts/2001/162/131140.pdf
4. Skaskiv O.B., Trusevych O.M. *Maximal term and the sum of a regularly convergent functional series*, Visn. Lviv. Un-ty, Ser. Mekh.-Math., **49** (1998), 75–79. (in Ukrainian)
5. Salo T.M., Skaskiv O.B. *The minimum modulus of gap power series and h-measure of exceptional sets*, arXiv:1512.05557v2 [math.CV], 17 Dec. 2015, 13 p. https://doi.org/10.48550/arXiv.1512.05557
6. Salo T.M., Skaskiv O.B. *Minimum modulus of lacunary power series and h-measure of exceptional sets*, Ufa Math. J., **9** (2017), №4, 135–144.

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