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SHARP ESTIMATES OF THE TOEPLITZ DETERMINANTS OF CERTAIN ORDER FOR THE PRIMARY SUBCLASSES OF UNIVALENT FUNCTIONS

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The primary object of this paper is to investigate sharp estimate to the Toeplitz determinants of third order for the class of bounded turning functions and fourth order for the class of starlike and convex functions in the open unit disk $\mathbb{D}$, which are the fundamental subclasses of univalent functions. The practical tools applied in the derivation of our main results are the coefficient inequalities for the analytic in $\mathbb{D}$ functions from the Carathéodory class.

The problem of finding sharp estimates to the Toeplitz determinants for the function f, when it is a member of certain subclass of univalent functions is technically difficult in the case when $q=4$ and $s\in\{1,2\}$ in (2), than that in the case when $q=3$ and $s\in\{1,2\}$. Here, in our present investigation, we have successfully derived the sharp bounds of third -order namely $T_{3,2}(f)$ for the class of Bounded turning functions and fourth-order Toeplitz determinants namely $T_{4,1}(f)$ and $T_{4,2}(f)$ for the class of starlike and convex functions. With the motivation of these results, researchers may obtain bounds (sharp) for other classes of analytic functions of higher order Toeplitz determinants.

1. Introduction and preliminaries. Let us denote by $\mathcal{H}$ the class of analytic functions in the open unit disk $\mathbb{D}=\{z\in\mathbb{C}\colon |z|<1\}$. By $\mathcal{A}$ we denote the class of functions $f\in\mathcal{H}$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n. \tag{1}$$

Let us also denote by $\mathcal{S}$ the class of univalent (i.e., one-to-one) functions $f\in\mathcal{A}$. A typical problem considered in geometric function theory is the problem of obtaining estimates of functionals defined both on the entire class of univalent functions and on its various subclasses such as subclasses of convex functions, starlike functions, etc. Among such functionals, the so-called Hankel determinants and Toeplitz determinants occupy an important place. In particular, Ch. Pommerenke [1] considered for given function $f\in\mathcal{S}$ of form (1) a Hankel determinant $H_{r,n}(f)$ for $r,\ n\in\mathbb{N}=\{1,2,3,...\}$.

For a subclass $\mathcal{F}$ of $\mathcal{A}$, with specific values of r and n, estimation of an upper bound of $H_{r,n}(f)$ with $f\in\mathcal{F}$ represents an exciting and significant problem, for instance (see [1–4]). The second Hankel determinant $H_{2,1}(f)=\begin{vmatrix}a_1 & a_2\\ a_2 & a_3\end{vmatrix}=a_3-a_2^2$ we obtain from Fekete-Szegö

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functional ([5]) $a_3-\rho a_2^2$ at $\rho=1$ (its estimates for some real ϱ and $f\in\mathcal{S}$ see also [17, Theorem 3.8]). Further, sharp bounds of the functional $H_{2,2}(f)=\begin{vmatrix}a_2 & a_3\\ a_3 & a_4\end{vmatrix}=a_2a_4-a_3^2$ ($r=2$, $n=2$), was considered by many authors for the classes of bounded-turning $\mathcal{R}$ (also called functions whose derivative have a positive real part), starlike $\mathcal{S}^*$ and convex $\mathcal{C}$ functions in the unit disc $\mathbb{D}$, were derived by Janteng et al. ([6, 7]) as 4/9, 1, and 1/8, respectively. The classes of functions $\mathcal{R}$, $\mathcal{S}^*$ and $\mathcal{C}$, are described by the conditions

$$\operatorname{Re} f'(z)>0,\quad \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>0\quad\text{and}\quad \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>0,$$

respectively. In recent years, special attention has been focussed research on the estimation of an upper bound (particularly sharp) to coefficient problems, namely $H_{3,1}(f)$ and $H_{4,1}(f)$ respectively for the functions in the class S and its subclasses, and more broadly in subclasses of class $\mathcal{A}$. For recent results and works on $H_{3,1}(f)$ and $H_{4,1}(f)$ (see [8–14]).

Closely related to Hankel determinants are the Toeplitz determinants. A review of the applications of Toeplitz matrices in a wide range of fields of pure and applied mathematics can be found in [15]. Recently, Thomas and Halim ([16]) introduced the concept of symmetric Toeplitz determinants for analytic functions f of the form (1) defined as follows

$$T_{q,s}(f)=\begin{vmatrix}a_s & a_{s+1} & \cdots & a_{s+q-1}\\ a_{s+1} & a_s & \cdots & a_{s+q-2}\\ \vdots & \vdots & \ddots & \vdots\\ a_{s+q-1} & a_{s+q-2} & \cdots & a_s\end{vmatrix} \tag{2}$$

for $q,\ s\in\mathbb{N}$. Now, many researchers are paying attention towards on the estimation of bounds (sharp) for Toeplitz determinants of specific order for the functions belong to certain subclasses of $\mathcal{A}$. In the present paper, we are making an attempt to estimate sharp bounds for the symmetric Toeplitz determinants of some orders for the functions f from the classes of bounded turning, starlike and convex functions.

Let $\mathcal{P}$ be a class of Carathéodory functions ([17]) g of the following form

$$g(z)=1+\sum_{n=1}^{+\infty}c_nz^n, \tag{3}$$

which have positive real part in $\mathbb{D}$. In view of the analytic conditions and (3), the coefficients of functions belongs to the subclasses namely $\mathcal{R}$, $\mathcal{S}^*$ and $\mathcal{C}$ of $\mathcal{S}$ have suitable representation expressed by coefficients of functions in the class $\mathcal{P}$.

2. A set of lemmas. The foundation for the proof of our main results, we need to apply the following lemmas and also to adopt some ideas from the work of Libera and Złotkiewicz ([18, 19]).

Lemma 1 ([20])**.** *If a function $g\in\mathcal{P}$ has form (3), then $|c_n|\le 2$, for all $n\in\mathbb{N}$, and the equality occurs for the function $h(z)=\frac{1+z}{1-z}$, $z\in\mathbb{D}$.*

Lemma 2 ([21])**.** *If a function $g\in\mathcal{P}$ has form (3), then*

$$|c_i-\mu c_jc_{i-j}|\le\begin{cases}2, & \text{if } \mu\in[0,1];\\ 2|2\mu-1|, & \text{if } \mu>1\end{cases}$$

for all $i,j\in\mathbb{N}$ such that $i>j$.

From Lemma 2, Livingston ([22]) proved that $|c_i - c_j c_{i-j}| \leq 2$. In paper [20] we find the following statement: if a function $g \in \mathcal{P}$ has form (3), then $|c_1^3 - 2c_1c_2 + c_3| \leq 2$.

Lemma 3 ([23])**.** *If a function $g \in \mathcal{P}$ has form* (3)*, then for all $J, K, L \in \mathbb{C}$*

$$|Jc_1^3 - Kc_1c_2 + Lc_3| \leq 2(|J| + |K - 2J| + |J - K + L|).$$

Lemma 4 ([24])**.** *If a function $g \in \mathcal{P}$ has form* (3)*, then $|c_r c_s - \lambda c_u c_v| \leq 4$ for all $r, s, u, v \in \mathbb{N}$ such that $r + s = u + v$, where $\lambda \in [0, 1]$.*

3. Sharp bounds for the third and fourth order Toeplitz determinants. In this section, we state and prove our main results associated with third and fourth-order Toeplitz determinants as follows.

Theorem 1. *If $f \in \mathcal{R}$, then $\left|T_{3,2}(f)\right| \leq \frac{25}{12}$, and the bound is best possible. The extremal function is $f(z) = z + iz^2 - \frac{2}{3}z^3 - \frac{i}{2}z^4 + \frac{2}{5}z^5 + \dots$, for which $f'(z) = \frac{1+iz}{1-iz}$.*

Proof. Let $f \in \mathcal{R}$ be of form (1). Then an analytic function $g = f'$ has form (3) and by condition $\operatorname{Re} f'(z) > 0$, we get $g \in \mathcal{P}$, So, from the equlity $g = f'$ by folmulas (1) and (3), we obtain

$$na_n = c_{n-1}, \quad n \geq 2. \tag{4}$$

Upon expanding the determinant for $T_{3,2}(f)$, substituting the values of a_2, a_3 and a_4 from (4), we obtain

$$T_{3,2}(f) := \frac{1}{288}\Big(36c_1^3 - 32c_1c_2^2 + 16c_3c_2^2 - 9c_1c_3^2\Big). \tag{5}$$

In order to apply Lemmas 1 and 2, expression in (5) can be reorganised as

$$T_{3,2}(f) := \frac{1}{288}\Big[-18c_1(c_2 - 2c_1^2) + 18c_1c_2(1 - c_2) - 14c_1c_2^2 + 16c_3c_2^2 - 9c_1c_3^2\Big]. \tag{6}$$

Taking modulus on both sides, and then applying the trangle inequality in (6), we get

$$|T_{3,2}(f)| \leq \frac{1}{288}\Big[18|c_1||c_2 - 2c_1^2| + 18|c_1||c_2||1 - c_2| + 14|c_1||c_2|^2 + 16|c_3||c_2|^2 + 9|c_1||c_3|^2\Big]. \tag{7}$$

Applying Lemma 1 and Lemma 2 in (7), it simplifies to $|T_{3,2}(f)| \leq 25/12$. □

Remark 2. For $f \in \mathcal{R}$, the bound obtained under Theorem 1 is sharp and it is more refined than the bound obtained by Firoz Ali et al. [25, see Theorem 2.12].

Theorem 3. *If $f \in \mathcal{S}^*$, then $\left|T_{4,1}(f)\right| \leq 116$ and the inequality is sharp. The extremal function is $f(z) = \frac{z}{(1-iz)^2} = z + 2iz^2 - 3z^3 - 4iz^4 + 5z^5 + \dots$.*

Proof. Let $f \in \mathcal{S}^*$, be of the form (1). Then there exists an analytic function $g \in \mathcal{P}$, such that

$$g(z) = \frac{zf'(z)}{f(z)} \quad (z \in \mathbb{D} \setminus \{0\}), \qquad g(0) = 1. \tag{8}$$

Substitute the values for f and g in (8), equating the coefficients, upon simplification, we obtain

$$a_2 = c_1;\ a_3 = \frac{c_2 + c_1^2}{2};\ a_4 = \frac{2c_3 + 3c_1c_2 + c_1^3}{6};\ a_5 = \frac{6c_4 + 8c_1c_3 + 3c_2^2 + 6c_1^2c_2 + c_1^4}{24}. \tag{9}$$

Using the values of a_n $(n=2,3,4)$ from (9), taking modulus on both sides and upon performing Lemmas 1, 2 and Lemma 3 appropriately, we obtain the following inequalities

$$
\begin{aligned}
|1-a_2^2| &= |1-c_1^2| \leq 1+|c_1|^2 = 1+4 = 5,\\
|a_2a_3-a_4| &= \frac{c_1}{2}|(c_2+c_1^2)-\frac{1}{6}(2c_3+3c_1c_2+c_1^3)| = \frac{1}{3}|c_1^3-c_3| \leq \frac{2}{3}(1+2+0) = 2,\\
|a_3-a_2a_4| &\leq \frac{1}{2}|c_2+c_1^2| + \frac{c_1}{6}|c_1^3+3c_1c_2+2c_3| = 3+8 = 11,\\
|a_2^2-a_3| &= |c_1^2-\frac{1}{2}(c_2+c_1^2) = \frac{1}{2}|c_2-c_1^2)| \leq \frac{1}{2}\cdot 2 = 1,\\
|a_2-a_2a_3| &= |c_1-\frac{c_1}{2}(c_2+c_1^2)| \leq |c_1|+\frac{|c_1|}{2}|c_2+c_1^2| = 2+2+4 = 8.
\end{aligned}
\tag{10}
$$

Upon expanding the determinant for $T_{4,1}(f)$, after simplifying, we get

$$
\begin{aligned}
T_{4,1}(f) &= [(a_2+a_3)^2-(1+a_2)(1+a_4)][(a_2-a_3)^2-(1-a_2)(1-a_4)]\\
&= (a_2^2+2a_2a_3-a_2a_4+a_3^2-a_2-a_4-1)\times(a_2^2-2a_2a_3-a_2a_4+a_3^2+a_2+a_4-1)\\
&= (1-a_2^2)^2+(a_2a_4-a_3^2)^2-(a_2-a_2a_3)^2-(a_2a_3-a_4)^2+2(a_3-a_2a_4)(a_2^2-a_3).
\end{aligned}
\tag{11}
$$

Now, taking modulus on both sides in the last equality, applying the traingle inequality and substituting the calculated values from (10) along with the result $|a_2a_4-a_3^2|\leq 1$ (see [6], upon simplification, we obtain $|T_{4,1}(f)|\leq 116$. □

Theorem 4. *If $f\in\mathcal{S}^*$, then $\left|T_{4,2}(f)\right|\leq 544$ and the inequality is sharp for the same function given in Theorem 3.*

Proof. Expanding the determinant for $T_{4,2}(f)$, after simplifying, we get

$$
\begin{aligned}
T_{4,2}(f) &= (a_2^2-a_2a_3-a_3^2+2a_3a_4-a_4^2-a_2a_5+a_3a_5)\\
&\quad\times(a_2^2+a_2a_3-a_3^2-2a_3a_4-a_4^2+a_2a_5+a_3a_5)\\
&= [(a_2-a_3)(a_2-a_5)-(a_3-a_4)^2]\times[(a_2+a_3)(a_2+a_5)-(a_3+a_4)^2].
\end{aligned}
\tag{12}
$$

Upon substituting the values of a_n $(n=2,3,4,5)$ from (9) in the expression $(a_2^2-a_2a_3-a_3^2+2a_3a_4-a_4^2-a_2a_5+a_3a_5)$, it simplifies to give

$$
\begin{aligned}
(a_2^2-a_2a_3-a_3^2+2a_3a_4-a_4^2-a_2a_5+a_3a_5) &= \frac{1}{144}\Big|-c_1^6+18c_1^5-3c_1^4c_2-36c_1^4\\
&\quad+60c_1^3c_2+8c_1^3c_3-9c_1^2c_2^2-72c_1^3-72c_1^2c_2+18c_1^2c_4+54c_1c_2^2\\
&\quad-24c_1c_2c_3+9c_2^3+144c_1^2-72c_1c_2-36c_1c_4-36c_2^2+48c_2c_3+18c_2c_4-16c_3^2\Big|\\
&= \frac{1}{144}\Big|-c_1^3(c_1^3+3c_1c_2-8c_3)+9c_1^2(c_4-c_2^2)+9c_1(c_1c_4-c_2c_3)+9c_2(c_2^2-c_1c_3)\\
&\quad+6c_2(c_4-c_1c_3)-16(c_3^2-\frac{3}{4}c_2c_4)+c_1^2c_2(60c_1-72)+c_1c_2(54c_2-72)+48(c_2c_3-\frac{3}{4}c_1c_4)\\
&\quad+18c_1^2(2-c_1)(4-c_1^2)-(36c_2^2-176)-176\Big|\\
&\leq |a_2^2-a_2a_3-a_3^2+2a_3a_4-a_4^2-a_2a_5+a_3a_5| \leq \frac{1}{144}[|c_1|^3|c_1^3+3c_1c_2-8c_3|\\
&\quad+9|c_1|^2|c_4-c_2^2|+9|c_1||c_1c_4-c_2c_3|+9|c_2||c_2^2-c_1c_3|+6|c_2||c_4-c_1c_3|\\
&\quad+16|c_3^2-\frac{3}{4}c_2c_4|+|c_1|^2|c_2||60c_1-72|+|c_1||c_2||54c_2-72|+48|c_2c_3-\frac{3}{4}c_1c_4|
\end{aligned}
$$

$$+18|c_1|^2|2-c_1|4-c_1^2|+|36c_2^2-176|+176]. \tag{13}$$

Applying Lemmas 1–4 in the inequality (13), it simplifies to give

$$|a_2^2-a_2a_3-a_3^2+2a_3a_4-a_4^2-a_2a_5+a_3a_5| \leq \frac{2448}{144}=17. \tag{14}$$

Similarly, substituting the values of a_n $(n=2,3,4,5)$ from (4) in the second multiplier $(a_2^2+a_2a_3-a_3^2-2a_3a_4-a_4^2+a_2a_5+a_3a_5)$, upon simplification, we obtain

$$\begin{aligned}
|a_2^2+a_2a_3-a_3^2-2a_3a_4-a_4^2+a_2a_5+a_3a_5| &= \frac{1}{144}\Big|-c_1^6-18c_1^5-3c_1^4c_2-36c_1^4\\
&-60c_1^3c_2+8c_1^3c_3-9c_1^2c_2^2+72c_1^3-72c_1^2c_2+18c_1^2c_4-54c_1c_2^2\\
&-24c_1c_2c_3+9c_2^3+144c_1^2+72c_1c_2+36c_1c_4-36c_2^2-48c_2c_3+18c_2c_4-16c_3^2\Big|\\
&=\frac{1}{144}\Big|-c_1^3(c_1^3+3c_1c_2-8c_3)+18c_1^2\Big(c_4-\frac{9}{18}c_2^2\Big)-24c_2(c_1c_3-c_2^2)\\
&+18\Big(c_2c_4-\frac{9}{24}c_2^3\Big)-18c_1^4(c_1-2)-48\Big(c_2c_3-\frac{36}{48}c_1c_4\Big)-60c_1^3c_2\\
&-54c_1c_2^2+72c_1^3+72c_1c_2+72c_1^2(2-c_2)-(36c_2^2-148)-148\Big|.
\end{aligned} \tag{15}$$

Applying the triangle inequality to (15) and Lemmas 1–4, we have

$$\begin{aligned}
|a_2^2+a_2a_3-a_3^2-2a_3a_4-a_4^2+a_2a_5+a_3a_5| &\leq \frac{1}{144}\Big[|c_1|^3|c_1^3+3c_1c_2-8c_3|\\
&+18|c_1|^2\Big|c_4-\frac{1}{2}c_2^2\Big|+24|c_2||c_1c_3-c_2^2|+18\Big|c_2c_4-\frac{3}{8}c_2^3\Big|+18|c_1|^4|c_1-2|\\
&+48\Big|c_2c_3-\frac{3}{4}c_1c_4\Big|+60|c_1|^3|c_2|+54|c_1||c_2|^2+72|c_1|^3+72|c_1||c_2|+72|c_1|^2|2-c_2|\\
&+|36c_2^2-148|+148\Big] \leq \frac{4608}{144}=32.
\end{aligned} \tag{16}$$

In view of inequalities (14) and (16), we obtain $\big|T_{4,2}(f)\big| \leq 17\cdot 32=544$. □

Theorem 5. *If $f\in\mathcal{C}$, then $\big|T_{4,1}(f)\big|\leq 8$ and the inequality is sharp. The extremal function is $f(z)=\frac{z}{1-iz}=z+iz^2-z^3-iz^4+z^5+\dots$.*

Proof. Let a fucntion $f\in\mathcal{C}$ be of the form (1). Then there exists an analytic function $g\in\mathcal{P}$, such that

$$f'(z)+zf''(z)=f'(z)g(z), \qquad z\in\mathbb{D}. \tag{17}$$

Substitute the values for f and g in (17), equating the coefficients, upon simplification, we obtain

$$a_2=\frac{c_1}{2};\ a_3=\frac{c_2+c_1^2}{6};\ a_4=\frac{2c_3+3c_1c_2+c_1^3}{24};\ a_5=\frac{6c_4+8c_1c_3+3c_2^2+6c_1^2c_2+c_1^4}{120}. \tag{18}$$

Substituting a_n, $(n=2,3,4)$ values from (18) in (11), after simplifying, we get

$$\begin{aligned}
T_{4,1}(f) &= \frac{1}{144}\big[c_1^4+18c_1^3-c_1^2c_2+36c_1^2+4c_2^2+6c_1c_2-6c_1c_3-72c_1-12c_3-144\big]\\
&\times\frac{1}{144}\big[c_1^4-18c_1^3-c_1^2c_2+36c_1^2+4c_2^2-6c_1c_2-6c_1c_3+72c_1+12c_3-144\big].
\end{aligned} \tag{19}$$

Upon rearranging the terms in the expression (19), in order to apply Lemmas specified in this paper, we have

$$
\begin{aligned}
|T_{4,1}(f)| = \frac{1}{144}\Big| & -36(4-c_1^2)+18c_1^3-8c_1(9-c_2)+c_1(c_1^3-c_3)-12c_3 \\
& +13\Big(c_2^2-\frac{4}{13}c_1c_3\Big)-c_1^2c_2-9c_2^2-2c_1c_2-c_1c_3\Big| \\
\times\frac{1}{144}\Big| & -18(2-c_1)(4-c_1^2)+12c_3-6c_1c_2-6c_1c_3+4\Big(c_2-\frac{1}{2}c_1^2\Big)^2 \\
+3c_1^2\Big(c_2-\frac{2}{3}c_1^2\Big)+2c_1^4\Big| & \leq \frac{1}{144}\Big[36|4-c_1^2|+18|c_1|^3+8|c_1||9-c_2|+|c_1||c_1^3-c_3|+12|c_3| \\
& +13\Big|c_2^2-\frac{4}{13}c_1c_3\Big|+|c_1|^2|c_2|+|9|c_2|^2+2|c_1||c_2|+|c_1||c_3|\Big] \\
& \times\frac{1}{144}\Big[18|2-c_1||4-c_1^2|+12|c_3|+6|c_1||c_2|+6|c_1||c_3|+4\Big|c_2-\frac{1}{2}c_1^2\Big|^2 \\
& +3|c_1|^2|c_2-\frac{2}{3}c_1^2|+2|c_1|^4\Big].
\end{aligned}
\tag{20}
$$

Performing Lemmas 1–4 in (20), upon simplification, we obtain $|T_{4,1}(f)| \leq 8$. □

Theorem 6. *If $f \in \mathcal{C}$, then $\big|T_{4,2}(f)\big| \leq 8$ and the inequality is sharp for the same function given in Theorem 5.*

Proof. Upon substituting the values of a_n, $(n = 2, 3, 4, 5)$ from (18) in one of the expression given in (12) of $T_{4,2}(f)$, it simplifies to

$$
\begin{aligned}
(a_2^2-a_2a_3-a_3^2+2a_3a_4-a_4^2-a_2a_5+a_3a_5) = & \frac{1}{2880}(-c_1^6+28c_1^5-2c_1^4c_2-80c_1^4 \\
& +88c_1^3c_2+12c_1^3c_3-9c_1^2c_2^2-240c_1^3-160c_1^2c_2-16c_1^2c_3+24c_1^2c_4+84c_1c_2^2 \\
& -28c_1c_2c_3+12c_2^3+720c_1^2-240c_1c_2-72c_1c_4-80c_2^2+80c_2c_3+24c_2c_4-20c_3^2).
\end{aligned}
\tag{21}
$$

Upon rearranging the terms in the equality (21), we have

$$
\begin{aligned}
\Big|a_2^2-a_2a_3-a_3^2+2a_3a_4-a_4^2-a_2a_5+a_3a_5\Big| = & \frac{1}{2880}\Big|-c_1^3(c_1^3+2c_1c_2-12c_3) \\
& +9c_1^2(c_4-c_2^2)+15c_1(c_1c_4-c_2c_3)+13c_2(c_4-c_1c_3)-20\Big(c_3^2-\frac{11}{20}c_2c_4\Big) \\
& +12c_2^2(c_2-1)+68c_2^2(c_1-1)-39c_1\Big(c_4-\frac{16}{39}c_2^2\Big)+80\Big(c_2c_3-\frac{33}{80}c_1c_4\Big) \\
& -16c_1^2(c_3-c_1c_2)-c_1c_2(240-72c_1^2)+c_1^3(28c_1^2-56)+c_1^2(320-184c_1) \\
& +c_1^2(160-80c_1^2)+c_1^2(240-160c_2)\Big| \leq \frac{1}{2880}\Big[|c_1|^3|c_1^3+2c_1c_2-12c_3| \\
& +9|c_1|^2|c_4-c_2^2|+15|c_1||c_1c_4-c_2c_3|+13|c_2||c_4-c_1c_3|+20\Big|c_3^2-\frac{11}{20}c_2c_4\Big| \\
& +12|c_2|^2|c_2-1|+68|c_2|^2|c_1-1|+39|c_1|\Big|c_4-\frac{16}{39}c_2^2\Big|+80\Big|c_2c_3-\frac{33}{80}c_1c_4\Big| \\
& +16|c_1|^2|c_3-c_1c_2|+|c_1||c_2||240-72c_1^2|+|c_1|^3|28c_1^2-56|+|c_1|^2|320-184c_1| \\
& +|c_1|^2|160-80c_1^2|+|c_1|^2|240-160c_2|\Big].
\end{aligned}
\tag{22}
$$

Applying Lemmas 1 – 4 in inequality (22), it simplifies to give

$$|a_2^2 - a_2a_3 - a_3^2 + 2a_3a_4 - a_4^2 - a_2a_5 + a_3a_5| \leq \frac{5760}{2880} = 2. \tag{23}$$

Similarly, upon substituting the values of a_n, $(n = 2, 3, 4, 5)$ from (18) in the other multiplier given in (12) of $T_{4,2}(f)$, it simplifies to

$$\begin{aligned}
&\Big|a_2^2 + a_2a_3 - a_3^2 - 2a_3a_4 - a_4^2 + a_2a_5 + a_3a_5) = \frac{1}{2880}(-c_1^6 - 28c_1^5 - 2c_1^4c_2 - 80c_1^4\\
&-88c_1^3c_2 + 12c_1^3c_3 - 9c_1^2c_2^2 + 240c_1^3 - 160c_1^2c_2 + 16c_1^2c_3 + 24c_1^2c_4 - 84c_1c_2^2\\
&-28c_1c_2c_3 + 12c_2^3 + 720c_1^2 + 240c_1c_2 + 72c_1c_4 - 80c_2^2 - 80c_2c_3 + 24c_2c_4 - 20c_3^2\Big|\\
&= \frac{1}{2880}\Big| - c_1^6 - 2c_1^4c_2 + 12c_1^3c_2 + 24c_1^2\Big(c_4 - \frac{3}{8}c_2^2\Big) + 24c_2\Big(c_4 - \frac{2}{3}c_1c_3\Big)\\
&+12c_2(c_2^2 - c_1c_3) + 72c_1(c_4 - c_2^2) - 88c_1^3c_2 + 16c_1\Big(c_1c_3 - \frac{3}{4}c_2^2\Big) - 28c_1^5 + 720c_1^2\\
&-80c_2c_3 - 80c_1^4 - 160c_1^2c_2 - 720c_1^2 + 240c_1^3 + 240c_1c_2 - 20c_3^2 - 368 + 368\Big|\\
&\leq \frac{1}{2880}\Big[|c_1|^6 + 2|c_1|^4|c_2| \ + 12|c_1|^3|c_2| + 24|c_1|^2\Big|c_4 - \frac{3}{8}c_2^2\Big| + 24\Big|c_2||c_4 - \frac{2}{3}c_1c_3\Big|\\
&+12|c_2||c_2^2 - c_1c_3| + 72|c_1||c_4 - c_2^2| + 16|c_1|\Big|c_1c_3 - \frac{3}{4}c_2^2\Big| + 28|c_1|^5 + 720|c_1|^2 + 80|c_2||c_3|\\
&+80|c_1|^2|9 - 2c_2| + 720|c_1|^2 + 240|c_1|^3 + 240|c_1||c_2| + |368 - 20c_3^2| + 368\Big].
\end{aligned} \tag{24}$$

Again applying Lemmas 1–4 in inequlity (24), it simplifies to give

$$|a_2^2 + a_2a_3 - a_3^2 - 2a_3a_4 - a_4^2 + a_2a_5 + a_3a_5| \leq \frac{11520}{2880} = 4 \tag{25}$$

In view of (12), from the inequlities (22) and (25), we obtain $\big|T_{4,2}(f)\big| \leq 8$. □

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