P. I. Boyko, H. P. Lopushanska

SOURCE INVERSE PROBLEM FOR FRACTIONAL KINETIC EQUATION

P. I. Boyko, H. P. Lopushanska. *Source inverse problem for fractional kinetic equation*, Mat. Stud. **64** (2025), 71–80.

We study the Cauchy problem for fractional kinetic equation in spaces with values in the entire scale of spaces of Bessel potentials, prove its unique, classical in time, solvability, obtaining an analogue of the maximum regularity of the solution according to Da Prato and Grisvard. We also study the inverse problem on determining a space-dependent component in the right-hand side of such equation under a time-integral additional condition.

We use the method of fundamental solution of the equation. We are investigating properties of the Green's operators of the Cauchy problem for fractional kinetic equation in spaces of continuous functions with values in spaces of Bessel potentials and find sufficient conditions for a time-local, classical in time and with values in spaces of Bessel potentials, unique solvability of the inverse problem. We define the unknown function on the right-hand side of the fractional kinetic equation over the entire scale of spaces of Bessel potentials.

The solution of the problem is reduced to the study of the unique solvability of some Fredholm's linear integral equation of the second kind in spaces of continuous functions with values in the entire scale of spaces of Bessel potentials. The unknown function on the right-hand side of the fractional kinetic equation is expressed through the solution of this integral equation.

1. Introduction. Fractional kinetic equation is used in various applied fields [1–3, 8, 22]. The Cauchy problem for the equation

$$u_t^{(\beta)} + a^2(-\Delta)^{\alpha/2}u = F(x,t), \quad (x,t) \in \mathbb{R}^n \times (0,T]$$

with the Riemann-Liouville derivative $u_t^{(\beta)}$ of order $\beta \in (0,1)$ and for the equation

$$D_t^\beta u + a^2(-\Delta)^{\alpha/2}u = F(x,t), \ (x,t) \in \mathbb{R}^n \times (0,T]$$

with the Caputo (Dzhrbasian-Nersessian-Caputo) fractional derivative

$$D_t^\beta u(x,t) = \frac{1}{\Gamma(1-\beta)}\left(\frac{\partial}{\partial t}\int_0^t \frac{u(x,\tau)}{(t-\tau)^\beta}d\tau - \frac{u(x,0)}{t^\beta}\right), \quad (x,t) \in \mathbb{R}^n \times [0,T]$$

of order $\beta \in (0,1)$ was studied by many authors, in particular, in [1, 4, 5, 13, 17], largest at $\alpha = 2$, a detailed bibliography on this case is given in [5, 9, 12, 15]. The unique solvability of the Cauchy problem and some inverse problems for such equations were obtained in certain subspaces of the Schwartz space distributions [10, 11, 18, 19] and in spaces of Bessel potentials [20].

In this paper, we prove the solvability of the Cauchy problem

$$D_t^\beta u + a^2(1-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}u = F(x,t), \ (x,t) \in \mathbb{R}^n \times (0,T], \quad u(x,0) = u_0(x), \ x \in R^n \quad (1)$$

2020 *Mathematics Subject Classification:* 35R11, 34K37.
Keywords: fractional derivative; fractional kinetic equation; inverse problem; integral overdetermination condition; convolution; Green's vector-function; Fourier transform.
doi:10.30970/ms.64.1.71-80

in the spaces of Bessel potentials in terms of spatial variables and classical with respect to time variable, obtaining an analogue of the maximum regularity of the solution according to Da Prato and Grisvard [14,21], and find sufficient conditions for a time-local unique solvability of a source inverse problem under a time-integral additional condition. Such problem was not studied. Here $(-\Delta)^{\alpha/2}$ and $(1-\Delta)^{\gamma/2}$ are determined using the Fourier transform

$$\mathcal{F}[(-\Delta)^{\alpha/2}\psi(x)] = |\xi|^{\alpha}\mathcal{F}[\psi(x)], \quad \mathcal{F}[(1-\Delta)^{\gamma/2}\psi(x)] = (1+|\xi|^2)^{\gamma/2}\mathcal{F}[\psi(x)].$$

2. Notation and definitions. Let $Q = \mathbb{R}^n \times (0,T]$, $n \in \mathbb{N}$, $S(\mathbb{R}^n)$ be a space of infinitely differentiable functions rapidly decreasing at infinity, $S'(\mathbb{R}^n)$ be a space of linear continuous functionals over $S(\mathbb{R}^n)$ (the space of distributions). Through $f*g$ we denote the convolution of f and g, $(f\widehat{*}g)(\tau) = \int_\tau^T f(t-\tau)g(t)dt$. We use the function

$$f_\lambda(t) = \begin{cases} \frac{\theta(t)t^{\lambda-1}}{\Gamma(\lambda)}, & \lambda > 0; \\ f'_{1+\lambda}(t), & \lambda \le 0, \end{cases}$$

where $\theta(t)$ is the Heaviside function, $\Gamma(\lambda)$ is the Gamma-function and note that

$$f_\lambda * f_\mu = f_{\lambda+\mu}, \quad f_\lambda \hat{*} f_\mu = f_{\lambda+\mu}.$$

Recall that the Riemann-Liouville derivative $v_t^{(\beta)}(x,t)$ of order $\beta > 0$ is determined by

$$v_t^{(\beta)}(x,t) = f_{-\beta}(t) * v(x,t),$$

and therefore $D_t^\beta v(x,t) = v_t^{(\beta)}(x,t) - \frac{v(x,0)}{\Gamma(1-\beta)t^\beta}$.

For $s \in \mathbb{R}$, $p > 1$ we determine the space of Bessel potentials ([23], p. 79)

$H^{s,p}(\mathbb{R}^n) = \{v \in S'(\mathbb{R}^n) \colon \|v\|_{s,p} = \|\mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}v]\|_{L_p(\mathbb{R}^n)} < +\infty\}$, and let

$$C\big([0,T]; H^{s,p}(\mathbb{R}^n)\big) = \{v(x,t) \colon \|v\|_{C\left([0,T];H^{s,p}(\mathbb{R}^n)\right)} = \max_{t\in[0,T]} \big\|v(\cdot,t)\big\|_{H^{s,p}(\mathbb{R}^n)} < +\infty\},$$

$$C_b\big((0,T]; H^{s,p}(\mathbb{R}^n)\big) = \{v(x,t) \colon \|v\|_{C_b\left((0,T];H^{s,p}(\mathbb{R}^n)\right)} = \sup_{t\in(0,T]} \big\|v(\cdot,t)\big\|_{H^{s,p}(\mathbb{R}^n)} < +\infty\},$$

$$C_{\alpha,\beta,\gamma}\big([0,T]; H^{s,p}(\mathbb{R}^n)\big) = \{v \in C\big([0,T]; H^{s+\alpha+\gamma,p}(\mathbb{R}^n)\big) \colon D_t^\beta v, (1-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2} v \in C_b\big((0,T]; H^{s,p}(\mathbb{R}^n)\big)\},$$

$$\|v\|_{C_{\alpha,\beta,\gamma}\left([0,T];H^{s,p}(\mathbb{R}^n)\right)} = \max\{\|v\|_{C\left([0,T];H^{s+\alpha+\gamma,p}(\mathbb{R}^n)\right)}, \|D_t^\beta v\|_{C_b\left((0,T];H^{s,p}(\mathbb{R}^n)\right)}, \|(1-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}v\|_{C_b\left((0,T];H^{s,p}(\mathbb{R}^n)\right)}\}.$$

Assumption (L): $\beta \in (0,1)$, $\alpha > \beta$, $\gamma \ge 0$.

Denote by $\tilde{C}_{\alpha,\beta,\gamma}(\bar{Q})$ a class of continuous functions $v(x,t)$, $(x,t) \in \bar{Q}$, equal to zero at $t \ge T$ and with continuous $(1-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}v$, $D_t^\beta v$ in Q.

The pair of functions $(G_0(x,t), G_1(x,t))$ is called ([5,17]) a *Green's vector function of the Cauchy problem* (1), if for rather regular and finite $F(x,t)$, $u_0(x)$, the function

$$u(x,t) = \int_0^t d\tau \int_{\mathbb{R}^n} G_0(x-y,t-\tau)F(y,\tau)dy + \int_{\mathbb{R}^n} G_1(x-y,t)u_0(y)dy, \quad (x,t) \in Q \quad (2)$$

is a classical (from $\tilde{C}_{\alpha,\beta,\gamma}(\bar{Q})$) solution of the problem (1).

The existence of a Green's vector-function was established, for example, in [1,4], and in [6,7,9,12,16] in the case $\gamma = 0$,

$$\mathcal{F}_{x\to\xi}[G_1(x,t)] := \mathcal{F}[G_1(x,t)] = \widehat{G}_1(\xi,t) = E_{\beta,1}(-a^2(1+|\xi|^2)^{\gamma/2}|\xi|^\alpha t^\beta),$$

where $E_{\beta,\mu}(z) = \sum\limits_{p=0}^{\infty} \frac{z^p}{\Gamma(p\beta+\mu)}$ is the Mittag-Leffler function.

For $\beta \in (0,1)$, $z > 0$ the function $E_{\beta,1}(-z) \in L_1(\mathbb{R}^n)$, $E_{\beta,\mu}(-z)$ is infinitely differentiable and compactly monotone if $\mu \geq \beta$: $(-1)^k(\frac{d}{dz})^k E_{\beta,\mu}(-z) \geq 0 \quad k = 0,1,2,\dots$, it has the bound

$$E_{\beta,\mu}(-z) \leq \frac{C}{1+z}, \quad C = \text{const} > 0.$$

Using the Fourier transform, we find G_0. We have the equation

$$(G_0)_t^{(\beta)} + a^2(1-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2} G_0 = \delta(x,t),\ (x,t) \in Q, \tag{3}$$

which, after the Fourier transformation in terms of spatial variables, takes the form

$$(\widehat{G}_0)_t^{(\beta)} + b^2 \widehat{G}_0 = \delta(t)$$

with $b^2 = b^2(\xi) = a^2(1+|\xi|^2)^{\gamma/2}|\xi|^{\alpha/2}$ and has the solution

$$\widehat{G}_0(\xi,t) = t^{\beta-1}E_{\beta,\beta}(-b^2t^\beta) = t^{\beta-1}E_{\beta,\beta}(-a^2(1+|\xi|^2)^{\gamma/2}|\xi|^\alpha t^\beta).$$

As in [17], for example, we show that

$$G_1(x,t) = f_{1-\beta}(t) * G_0(x,t), \quad (x,t) \in Q.$$

3. Solvability of the direct problem. Taking into account properties of the convolution and components of the Green's vector-function we obtain the solvability of the Cauchy problem (1) throughout the scale of spaces $H^{s,p}(\mathbb{R}^n)$ in space variables.

Lemma 1. *For $\varrho \leq \alpha+\gamma$ and each $t \in (0,T]$ the functions*

$$g_j(\xi,t,\varrho) = (1+|\xi|^2)^{\frac{\varrho}{2}}\mathcal{F}[G_j](\xi,t), \quad j = 0,1,$$

are bounded and continuous in space variables $\xi \in \mathbb{R}^n$. There exist positive constants $d_j = d_j(p)$ such that for all $p > 1$, $f \in L_p(\mathbb{R}^n)$

$$\|\mathcal{F}^{-1}[g_j(\xi,t,\varrho)\mathcal{F}[f]]\|_{L_p(\mathbb{R}^n)} \leq d_j w_j(t)\|f\|_{L_p(\mathbb{R}^n)}, \quad t \in (0,T], \quad j = 0,1, \tag{4}$$

where $w_0(t) = \max\{t^{\beta-1}, t^{\beta(1-\frac{\varrho}{\alpha+\gamma})-1}\}$, $w_1(t) = \max\{1, t^{-\frac{\beta\varrho}{\alpha+\gamma}}\}$ for each $t \in (0,T]$.

Proof. The lemma is proved according to the scheme of the proof of Lemma 1 from [13], where the case $\gamma = 0$ was considered. □

Theorem 1. *Let the assumption (L) holds, $\beta,\theta \in (0,1)$, $1 < p < \frac{1}{1-\beta\theta}$, $s \in \mathbb{R}$, $F \in C_b\big((0,T]; H^{s+(\alpha+\gamma)\theta,p}(\mathbb{R}^n)\big)$, $u_0 \in H^{s+\alpha+\gamma,p}(\mathbb{R}^n)$. Then there exists the unique solution*

$$u(x,t) = F(x,t) * G_0(x,t) + u_0(x) * G_1(x,t), \qquad (x,t) \in Q, \tag{5}$$

of the problem (1) in the space $C_{\alpha,\beta,\gamma}\big([0,T]; H^{s,p}(\mathbb{R}^n)\big)$ and the bound

$$\|u\|_{C_{\alpha,\beta,\gamma}\left([0,T];H^{s,p}(\mathbb{R}^n)\right)} \leq b_0\|F\|_{C_b\left((0,T];H^{s+(\alpha+\gamma)\theta,p}(\mathbb{R}^n)\right)} + b_1\|u_0\|_{H^{s+\alpha+\gamma,p}(\mathbb{R}^n)} \tag{6}$$

holds where b_0, b_1 are positive constants.

Proof. Similarly to [13], at the beginning, we show the existence of convolutions from (5) in the space $C\big([0,T]; H^{s+\alpha+\gamma,p}(\mathbb{R}^n)\big)$. We have

$$\mathcal{F}^{-1}\left[(1+|\xi|^2)^{\frac{s+\alpha+\gamma}{2}}\mathcal{F}[u_0(\cdot) * G_1(\cdot,t)]\right] = \mathcal{F}^{-1}\left[\mathcal{F}[G_1](\xi,t)\big(1+|\xi|^2\big)^{\frac{s+\alpha+\gamma}{2}}\mathcal{F}[u_0](\xi)\right] =$$

$$= \mathcal{F}^{-1}\left[g_1(\xi,t,0)\mathcal{F}\big[\mathcal{F}^{-1}[\big(1+|\xi|^2\big)^{\frac{s+\alpha+\gamma}{2}}\mathcal{F}[u_0](\xi)]\big]\right].$$

By assumption, $\mathcal{F}^{-1}\big[\big(1+|\xi|^2\big)^{\frac{s+\alpha+\gamma}{2}}\mathcal{F}[u_0](\xi)\big] \in L_p(\mathbb{R}^n)$, by Lemma 1 the function $g_1(\xi,t,0)$ is a multiplicator in $L_p(\mathbb{R}^n)$ in variables ξ. Therefore,

$$\big\|u_0 * G_1\big\|_{H^{s+\alpha+\gamma,p}(\mathbb{R}^n)} = \left\|\mathcal{F}^{-1}\left[(1+|\xi|^2)^{\frac{s+\alpha+\gamma}{2}}\mathcal{F}[u_0(\cdot)*G_1(\cdot,t)]\right]\right\|_{L_p(\mathbb{R}^n)} \le$$
$$\le c_1 w_1(t,0)\left\|\mathcal{F}^{-1}\big[(1+|\xi|^2)^{\frac{s+\alpha+\gamma}{2}}\mathcal{F}[u_0](\xi)\big]\right\|_{L_p(\mathbb{R}^n)} = c_1\|u_0\|_{H^{s+\alpha+\gamma}(\mathbb{R}^n)}.$$

We took into account that $w_1(t,0)=1$. From the inequalities obtained above, for all $s\in\mathbb{R}$, $p>1$, $t\in[0,T]$, we obtained the existence of a convolution $u_0 * G_1 \in C\big([0,T];H^{s+\alpha+\gamma,p}(\mathbb{R}^n)\big)$ and the bound

$$\big\|u_0 * G_1\big\|_{C\left([0,T];H^{s+\alpha+\gamma,p}(\mathbb{R}^n)\right)} \le c_1\big\|u_0\big\|_{H^{s+\alpha+\gamma,p}(\mathbb{R}^n)}. \tag{7}$$

Hereinafter $c_i,\ i\in\mathbb{N}$ are positive constants.

For each $t\in[0,T]$, let us estimate the norm

$$\big\|(F*G_0)(\cdot,t)\big\|_{H^{s+\alpha+\gamma,p}(\mathbb{R}^n)} = \left\|\mathcal{F}^{-1}\left[(1+|\xi|^2)^{\frac{s+\alpha+\gamma}{2}}\mathcal{F}[F*G_0]\right]\right\|_{L_p(\mathbb{R}^n)} =$$
$$= \left\|\mathcal{F}^{-1}\left[(1+|\xi|^2)^{\frac{s+\alpha+\gamma}{2}}\mathcal{F}\left[\int_0^t F(\cdot,\tau)*G_0(\cdot,t-\tau)\,d\tau\right]\right]\right\|_{L_p(\mathbb{R}^n)} \le$$
$$\le t^{1-\frac{1}{p}}\left\{\int_0^t d\tau\int_{\mathbb{R}^n}\big|h_0(x,t,\tau)\big|^p\,dx\right\}^{1/p},$$

where

$$h_0(\cdot,t,\tau) = \mathcal{F}^{-1}\left[(1+|\xi|^2)^{\frac{s+\alpha+\gamma}{2}}\mathcal{F}[F(\cdot,\tau)*G_0(\cdot,t-\tau)]\right] =$$
$$= \mathcal{F}^{-1}\left[\big(1+|\xi|^2\big)^{\frac{(\alpha+\gamma)(1-\theta)}{2}}\mathcal{F}[G_0](\xi,t-\tau)\big(1+|\xi|^2\big)^{\frac{s+(\alpha+\gamma)\theta}{2}}\mathcal{F}[F](\xi,\tau)\right] =$$
$$= \mathcal{F}^{-1}\left[g_0(\xi,t-\tau,(\alpha+\gamma)(1-\theta))\mathcal{F}\big[\mathcal{F}^{-1}[\big(1+|\xi|^2\big)^{\frac{s+(\alpha+\gamma)\theta}{2}}\mathcal{F}[F](\xi,\tau)]\big]\right].$$

By Lemma 1, the function $g_0(\xi,t-\tau,(\alpha+\gamma)(1-\theta))$ is a multiplicator in $L_p(\mathbb{R}^n)$ in variables ξ. By assumption,

$$\mathcal{F}^{-1}\left[\big(1+|\xi|^2\big)^{\frac{s+(\alpha+\gamma)\theta}{2}}\mathcal{F}[F](\xi,\tau)\right] \in L_p(\mathbb{R}^n) \text{ for each } \tau\in[0,T].$$

Then

$$\big\|h_0(\cdot,t,\tau)\big\|_{L_p(\mathbb{R}^n)} \le c_2 w_0(t-\tau,(\alpha+\gamma)(1-\theta))\left\|\mathcal{F}^{-1}\big[(1+|\xi|^2)^{\frac{s+(\alpha+\gamma)\theta}{2}}\mathcal{F}[F](\xi,\tau)\big]\right\|_{L_p(\mathbb{R}^n)} =$$
$$= c_2 w_0(t-\tau,(\alpha+\gamma)(1-\theta))\,\|F(\cdot,\tau)\|_{H^{s+(\alpha+\gamma)\theta,p}(\mathbb{R}^n)} \quad \forall\tau\in(0,T].$$

From previous transformations, we get

$$\big\|(F*G_0)(\cdot,t)\big\|_{H^{s+\alpha+\gamma,p}(\mathbb{R}^n)} \le$$
$$\le c_2 t^{1-\frac{1}{p}}\left\{\int_0^t w_0^p(t-\tau,(\alpha+\gamma)(1-\theta))\,d\tau\right\}^{1/p}\|F\|_{C_b\left((0,T];H^{s+(\alpha+\gamma)\theta,p}(\mathbb{R}^n)\right)},$$

and under the condition $p(1-\beta\theta) < 1$ which is the condition for the existence of the integral

$$\int_0^t w_0^p(t-\tau,(\alpha+\gamma)(1-\theta))\,d\tau,$$

for all $t \in [0,T]$, $s \in \mathbb{R}$ we obtain the existence of the convolution $\int_0^t F(\cdot,\tau) * G_0(\cdot,t-\tau)\,d\tau$ in $H^{s+\alpha+\gamma,p}(\mathbb{R}^n)$ and the bound

$$\left\|\int_0^t F(\cdot,\tau) * G_0(\cdot,t-\tau)\,d\tau\right\|_{H^{s+\alpha+\gamma,p}(\mathbb{R}^n)} \le c_3 t^{\beta\theta}\|F\|_{C_b\left((0,T];H^{s+(\alpha+\gamma)\theta,p}(\mathbb{R}^n)\right)}.$$

Therefore, $F * G_0 \in C\big([0,T];H^{s+\alpha+\gamma,p}(\mathbb{R}^n)\big)$ and, considering (7), we get the estimate

$$\|u\|_{C\left([0,T];H^{s+\alpha+\gamma,p}(\mathbb{R}^n)\right)} \le c_1\|u_0\|_{H^{s+\alpha+\gamma,p}(\mathbb{R}^n)} + c_3 T^{\beta\theta}\|F\|_{C_b\left((0,T];H^{s+(\alpha+\gamma)\theta,p}(\mathbb{R}^n)\right)}. \tag{8}$$

Because

$$\mathcal{F}^{-1}\left[(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}[(1-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}u]\right] = \mathcal{F}^{-1}\left[\left(1+|\xi|^2\right)^{\frac{s+\gamma}{2}}|\xi|^\alpha\mathcal{F}[u]\right] =$$
$$= \mathcal{F}^{-1}\left[\frac{|\xi|^\alpha}{\left(1+|\xi|^2\right)^{\frac{\alpha}{2}}}\left(1+|\xi|^2\right)^{\frac{s+\alpha+\gamma}{2}}\mathcal{F}[u]\right],$$

as proven, $\mathcal{F}^{-1}\left[(1+|\xi|^2)^{\frac{s+\alpha+\gamma}{2}}\mathcal{F}[u]\right] \in L_p(\mathbb{R}^n)$ for each $t \in [0,T]$, and the function $\frac{|\xi|^\alpha}{(1+|\xi|^2)^{\frac{\alpha}{2}}}$ is a multiplicator in $L_p(\mathbb{R}^n)$, then for all $t \in (0,T]$ we get

$$\left\|\mathcal{F}^{-1}\left[(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}[(1-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}u]\right]\right\|_{L_p(\mathbb{R}^n)} \le c_4\left\|\mathcal{F}^{-1}\left[(1+|\xi|^2)^{\frac{s+\alpha+\gamma}{2}}\mathcal{F}[u]\right]\right\|_{L_p(\mathbb{R}^n)}.$$

We obtained that for the function (5) (of class $C\big([0,T];H^{s+\alpha+\gamma,p}(\mathbb{R}^n)\big)$ as shown) the condition $(1-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}u \in C_b\big((0,T];H^{s,p}(\mathbb{R}^n)\big)$ is also met and the following bound holds

$$\|(1-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}u\|_{C_b\left((0,T];H^{s,p}(\mathbb{R}^n)\right)} \le c_4\|u\|_{C\left([0,T];H^{s+\alpha+\gamma,p}(\mathbb{R}^n)\right)}. \tag{9}$$

It follows from the equation (1) that

$${}^C D_t^\beta u = -a^2(1-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}u + F \in C_b\big((0,T];H^{s,p}(\mathbb{R}^n)\big)$$

and the following bound holds

$$\|{}^C D_t^\beta u\|_{C_b\left((0,T];H^{s,p}(\mathbb{R}^n)\right)} \le \|a^2(1-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}u\|_{C\left([0,T];H^{s,p}(\mathbb{R}^n)\right)} + \|F\|_{C_b\left((0,T];H^{s,p}(\mathbb{R}^n)\right)} \le$$
$$\le c_5\|u\|_{C\left([0,T];H^{s+\alpha+\gamma,p}(\mathbb{R}^n)\right)} + c_6\|F\|_{C_b\left((0,T];H^{s+(\alpha+\gamma)\theta,p}(\mathbb{R}^n)\right)}.$$

From this and bounds (8), (9) the estimate (6) follows.

We have shown that the function (5) belongs to $C_{\alpha,\beta,\gamma}\big([0,T];H^{s,p}(\mathbb{R}^n)\big)$. By properties of the Green's vector-function it is the solution of the Cauchy problem (1). □

4. Solution of the inverse problem. We find sufficient conditions for existence and uniqueness of the solution (u,g) of the inverse problem

$$D_t^{\beta}u+(1-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}u=g(x)F_0(t)+F(x,t),\quad (x,t)\in\mathbb{R}^n\times(0,T]:=Q, \tag{10}$$

$$u(x,0)=u_0(x),\quad x\in\mathbb{R}^n, \tag{11}$$

$$\frac{1}{T}\int_0^T u(x,t)dt=\Phi(x),\quad x\in\mathbb{R}^n \tag{12}$$

where F,F_0,u_0,Φ are the given functions.

For $\theta\in(0,1)$ the pair $(u,g)\in C_{\alpha,\beta,\gamma}\big([0,T];H^{s,p}(\mathbb{R}^n)\big)\times H^{s+(\alpha+\gamma)\theta,p}(\mathbb{R}^n)$ is called a solution of the problem (10)–(12) if it satisfies the equation (10) and the conditions (11), (12).

Note that

$$\int_0^T(1-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}u(x,t)dt=(1-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}\int_0^T u(x,t)dt=(1-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}\Phi(x),$$

$$\int_0^T D_t^{\beta}u(x,t)dt=\int_0^T[f_{1-\beta}(t)*u_t(x,t)]dt=\int_0^T\Big(\int_0^t f_{1-\beta}(t-\tau)u_\tau(x,\tau)d\tau\Big)dt=$$
$$=\int_0^T u_\tau(x,\tau)\Big(\int_\tau^T f_{1-\beta}(t-\tau)dt\Big)d\tau=\int_0^T u_\tau(x,\tau)f_{2-\beta}(T-\tau)d\tau=$$
$$=-u_0(x)f_{2-\beta}(T)+\int_0^T u(x,\tau)f_{1-\beta}(T-\tau)d\tau.$$

Applying the condition (12) to both sides of the equation (10) we get

$$\frac{1}{T}\Big[-u_0(x)f_{2-\beta}(T)+\int_0^T u(x,\tau)f_{1-\beta}(T-\tau)d\tau\Big]+(1-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}\Phi(x)=$$
$$=\frac{1}{T}\Big[g(x)\int_0^T F_0(\tau)d\tau+\int_0^T F(x,\tau)d\tau\Big],\quad x\in\mathbb{R}^n.$$

Assuming $N=N(T):=\frac{1}{T}\int_0^T F_0(t)dt\neq 0$, we find the following expression for the unknown function

$$g(x)=\frac{1}{N}(1-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}\Phi(x)-$$
$$-\frac{1}{NT}\Big[u_0(x)f_{2-\beta}(T)-\int_0^T u(x,\tau)f_{1-\beta}(T-\tau)d\tau+\int_0^T F(x,\tau)d\tau\Big],\quad x\in\mathbb{R}^n. \tag{13}$$

Lemma 2. *Let $s\in\mathbb{R},p>1,u,F\in C_b\big((0,T];H^{s,p}(\mathbb{R}^n)\big)$, $u_0\in H^{s,p}(\mathbb{R}^n)$, $\Phi\in H^{s+\alpha+\gamma,p}(\mathbb{R}^n)$, $F_0\in C[0,T]$ and $\int_0^T F_0(\tau)d\tau\neq 0$. Then the function g, defined by (13), belongs to $H^{s,p}(\mathbb{R}^n)$ and the following estimate holds*

$$\|g\|_{H^{s,p}(\mathbb{R}^n)}\le a_0\|u\|_{C\big([0,T];H^{s,p}(\mathbb{R}^n)\big)}+$$
$$+a_1\|F\|_{C_b\big((0,T];H^{s,p}(\mathbb{R}^n)\big)}+a_2\|u_0\|_{H^{s,p}(\mathbb{R}^n)}+a_3\|\Phi\|_{H^{s+\alpha+\gamma,p}(\mathbb{R}^n)},\quad t\in[0,T]$$

where a_0,a_1,a_2,a_3 are positive constants.

Proof. For all $s\in\mathbb{R},\ p>1$ we have

$$\|(1-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}\Phi\|_{H^{s,p}(\mathbb{R}^n)}=\big\|\mathcal{F}^{-1}\big[(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}[(1-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}\Phi](\xi)\big]\big\|_{L_p(\mathbb{R}^n)}=$$
$$=\big\|\mathcal{F}^{-1}\big[|\xi|^{\alpha}(1+|\xi|^2)^{\frac{s+\gamma}{2}}\mathcal{F}[\Phi](\xi)\big]\big\|_{L_p(\mathbb{R}^n)}=\big\|\mathcal{F}^{-1}\big[\frac{|\xi|^{\alpha}}{(1+|\xi|^2)^{\frac{\alpha}{2}}}(1+|\xi|^2)^{\frac{s+\alpha+\gamma}{2}}\mathcal{F}[\Phi](\xi)\big]\big\|_{L_p(\mathbb{R}^n)}\le$$

$$\le C\big\|\mathcal{F}^{-1}\big[(1+|\xi|^2)^{\frac{s+\alpha+\gamma}{2}}\mathcal{F}[\Phi](\xi)\big]\big\|_{L_p(\mathbb{R}^n)} = C\big\|\Phi\big\|_{H^{s+\alpha+\gamma,p}(\mathbb{R}^n)},$$

and then the required estimate follows from the formula (13). □

Assumptions:

(A) $\beta \in [\frac{1}{2},1)$, $1 < p < \frac{1}{\beta}$, $\alpha > \beta$, $\gamma \ge 0$, $s \in \mathbb{R}$, $F \in C_b\big((0,T];H^{s+\alpha+\gamma,p}(\mathbb{R}^n)\big)$, $u_0 \in H^{s+\alpha+\gamma,p}(\mathbb{R}^n)$, $\Phi \in H^{s+2(\alpha+\gamma),p}(\mathbb{R}^n)$, $F_0 \in C[0,T]$, $\int_0^T F_0(t)dt \ne 0$;

(B) $\frac{T\max\limits_{\tau\in[0,T]}|F_0(\tau)|}{|\int_0^T F_0(t)dt|}$ is bounded by an increasing function of T.

Note that the assumption (B) is fulfilled, for example, for $F_0(t)$ positive monotone increasing and with $F_0(0) \ne 0$.

Note that $H^{s+\alpha+\gamma,p}(\mathbb{R}^n) \subset H^{s+(\alpha+\gamma)\theta,p}(\mathbb{R}^n)$ for each $\theta \in (0,1)$, in particular, such that $\frac{1}{\beta} \le \frac{1}{1-\beta\theta}$, that is $\theta \in [\frac{1-\beta}{\beta},1)$ for $\beta \in [\frac{1}{2},1)$. It follows from Lemma 2 that in assumption (A), defined by (13) function $g \in H^{s+\alpha+\gamma,p}(\mathbb{R}^n) \subset H^{s+(\alpha+\gamma)\theta,p}(\mathbb{R}^n)$ and the following estimate holds

$$\|g\|_{H^{s+(\alpha+\gamma)\theta,p}(\mathbb{R}^n)} \le b\|g\|_{H^{s^*,p}(\mathbb{R}^n)} \le b_0\big\|u\big\|_{C\big([0,T];H^{s^*,p}(\mathbb{R}^n)\big)} + b_1\big\|F\big\|_{C_b\big((0,T];H^{s^*,p}(\mathbb{R}^n)\big)} +$$
$$+ b_2\|u_0\|_{H^{s^*,p}(\mathbb{R}^n)} + b_3\|\Phi\|_{H^{s+2(\alpha+\gamma),p}(\mathbb{R}^n)}, \quad t \in [0,T]$$

where $s^* = s+\alpha+\gamma$, and b, b_0, b_1, b_2, b_3 are positive constants dependent on T. Then in assumption (A), by Theorem 1, there exists the unique solution u of the Cauchy problem (10), (11), $u \in C_{\alpha,\beta,\gamma}\big([0,T];H^{s,p}(\mathbb{R}^n)\big)$ and it is defined by

$$u(x,t) = \big(g(x)F_0(t) + F(x,t)\big) * G_0(x,t) + u_0(x) * G_1(x,t), \quad (x,t) \in \bar{Q}. \tag{14}$$

Substituting (13) into the formula (14), we obtain

$$u(x,t) = \int_0^t \Big\{\frac{F_0(\tau)}{N}\Big[(1-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}\Phi(x) - \frac{1}{T}\Big(u_0(x)f_{2-\beta}(T) -$$
$$- \int_0^T u(x,\eta)f_{1-\beta}(T-\eta)d\eta\Big) + \frac{1}{T}\int_0^T F(x,\eta)d\eta\Big]\Big\} * G_0(x,t-\tau)\,d\tau + u_0(x) * G_1(x,t)$$

for $(x,t) \in Q$, i.e. the Fredholm's linear integral equation of the second kind

$$u(x,t) - \frac{1}{NT}\int_0^T \Big[u(x,\eta) * \int_0^t F_0(\tau)G_0(x,t-\tau)d\tau\Big]f_{1-\beta}(T-\eta)d\eta = v_0(x,t), \ \ (x,t) \in Q, \tag{15}$$

where

$$v_0(x,t) = \int_0^t \frac{F_0(\tau)}{N}(1-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}\Phi(x) * G_0(x,t-\tau)d\tau -$$
$$-\frac{1}{NT}\int_0^t \Big[F_0(\tau)u_0(x)f_{2-\beta}(T) + \int_0^T F(x,\eta)d\eta\Big] * G_0(x,t-\tau)d\tau + u_0(x) * G_1(x,t) \tag{16}$$

for $(x,t) \in Q$, and here the symbol $*$ means the convolution in space variables.

Theorem 2. *Under assumptions (A), (B) there exists $T_1 > 0$ and for each $T \in [0,T_1]$ the unique solution*

$$(u,g) \in C_{\alpha,\beta,\gamma}\big([0,T];H^{s,p}(\mathbb{R}^n)\big) \times H^{s+\alpha+\gamma,p}(\mathbb{R}^n) \subset C_{\alpha,\beta,\gamma}\big([0,T];H^{s,p}(\mathbb{R}^n)\big) \times H^{s+(\alpha+\gamma)\theta,p}(\mathbb{R}^n)$$

of the problem (10)–(12): g is defined by (13) where u is the solution of the equation (15).

Proof. It was shown above that under conditions of this theorem, the solution u of the Cauchy problem (10), (11) satisfies the equation (15) in $C\big([0,T];H^{s+\alpha+\gamma,p}(\mathbb{R}^n)\big)$. On the contrary, each solution $u\in C\big([0,T];H^{s+\alpha+\gamma,p}(\mathbb{R}^n)\big)$ of the equation (15) satisfies this Cauchy problem, because the equation (15) is obtained with (14) by substituting a specific function g from (13), and according to Theorem 1, $u\in C_{\alpha,\beta,\gamma}\big([0,T];H^{s,p}(\mathbb{R}^n)\big)$. The solution $u\in C\big([0,T];H^{s+\alpha+\gamma,p}(\mathbb{R}^n)\big)$ of the equation (15) also satisfies the condition (12) if g is defined by (13). Indeed, assuming that this is not the case, and

$$\frac{1}{T}\int_0^T u(x,t)dt=\Phi^*(x),\quad x\in\mathbb{R}^n$$

with some $\Phi^*\in H^{s+2(\alpha+\gamma),p}(\mathbb{R}^n)\subset H^{s+\alpha+\gamma,p}(\mathbb{R}^n)$, by the uniqueness of the solution of the Cauchy problem, from here and from formulas (12), (13) we obtain

$$(1-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}\big(\Phi^*(x)-\Phi(x)\big)=0 \iff (1+|\xi|^2)^{\gamma/2}|\xi|^{\alpha}\,\big(\mathcal{F}[\Phi^*](\xi)-\mathcal{F}[\Phi](\xi)\big)=0,$$

and therefore, $\Phi^*(x)-\Phi(x)=0$, $x\in\mathbb{R}^n$.

It remains to prove the solvability of the equation (15) in $C\big([0,T];H^{s+\alpha+\gamma,p}(\mathbb{R}^n)\big)$ and the uniqueness of the solution of the inverse problem.

As in the proof of Theorem 1 we show that $v_0\in C\big([0,T];H^{s+\alpha+\gamma,p}(\mathbb{R}^n)\big)$.

On $C\big([0,T];H^{s+\alpha+\gamma,p}(\mathbb{R}^n)\big)$ we introduce the operator P

$$(Pv)(x,t)=\frac{1}{NT}\int_0^T\Big[v(x,\eta)*\int_0^t F_0(\tau)G_0(x,t-\tau)d\tau\Big]f_{1-\beta}(T-\eta)d\eta+v_0(x,t),$$
$$(x,t)\in\bar{Q},\ v\in C\big([0,T];H^{s+\alpha+\gamma,p}(\mathbb{R}^n)\big).$$

For $v\in C\big([0,T];H^{s+\alpha+\gamma,p}(\mathbb{R}^n)\big)$, taking into account the above estimates and Lemma 1, we get

$$\|(Pv)(\cdot,t)\|_{H^{s^*,p}(\mathbb{R}^n)}\le\|v_0(\cdot,t)\|_{H^{s^*,p}(\mathbb{R}^n)}+$$
$$+\frac{1}{NT}\Big\|\int_0^T v(\cdot,\eta)*\int_0^t F_0(\tau)G_0(\cdot,t-\tau)\,d\tau\,f_{1-\beta}(T-\eta)\,d\eta\Big\|_{H^{s^*,p}(\mathbb{R}^n)};$$
$$\Big\|\int_0^T v(\cdot,\eta)*\int_0^t F_0(\tau)G_0(\cdot,t-\tau)\,d\tau\,f_{1-\beta}(T-\eta)\,d\eta\Big\|_{H^{s^*,p}(\mathbb{R}^n)}=$$
$$=\Big\|\mathcal{F}^{-1}\Big[(1+|\xi|^2)^{\frac{s^*}{2}}\mathcal{F}\Big[\int_0^T f_{1-\beta}(T-\eta)\,d\eta\int_0^t F_0(\tau)v(\cdot,\eta)*G_0(\cdot,t-\tau)\,d\tau\Big]\Big]\Big\|_{L_p(\mathbb{R}^n)}\le$$
$$\le t^{1-\frac{1}{p}}\max_{\tau\in[0,T]}|F_0(\tau)|\times$$
$$\times\Big\{\int_0^t d\tau\Big|\int_0^T f_{1-\beta}(T-\eta)d\eta\int_{\mathbb{R}^n}\mathcal{F}^{-1}\Big[(1+|\xi|^2)^{\frac{s^*}{2}}\mathcal{F}[v(x,\eta)]\mathcal{F}[G_0(x,t-\tau)]\Big]\Big|^p dx\Big\}^{1/p}\le$$
$$\le c_7T^{1-\beta}t^{1-\frac{1}{p}}\max_{\tau\in[0,T]}|F_0(\tau)|\Big[\int_0^t w_0^p(t-\tau,0)d\tau\Big]^{\frac{1}{p}}\ \|v\|_{C\left([0,T];H^{s^*,p}(\mathbb{R}^n)\right)},$$

where $s^*=s+\alpha+\gamma$. Hence,

$$\|Pv\|_{C\left([0,T];H^{s^*,p}(\mathbb{R}^n)\right)}\le\|v_0\|_{C\left([0,T];H^{s^*,p}(\mathbb{R}^n)\right)}+\frac{c_8T\max\limits_{\tau\in[0,T]}|F_0(\tau)|}{|\int_0^T F_0(\tau)d\tau|}\|v\|_{C\left([0,T];H^{s^*,p}(\mathbb{R}^n)\right)}.$$

So, $P\colon C\big([0,T];H^{s+\alpha+\gamma,p}(\mathbb{R}^n)\big)\to C\big([0,T];H^{s+\alpha+\gamma,p}(\mathbb{R}^n)\big)$. The Neumann series for the solution of the equation (15) converges under the condition (B) for $T<T_1$ with some $T_1>0$.

Let us prove the uniqueness of the solution of the inverse problem. Let (u_1,g_1), (u_2,g_2) be two solutions of the problem (10)–(12). Then for $u=u_1-u_2$, $g=g_1-g_2$ we have the problem

$$D_t^{\beta}u+a^2(1-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}u=g(x)F_0(t),\quad (x,t)\in Q,$$

$$u(x,0)=0,\quad x\in\mathbb{R}^n,\qquad \frac{1}{T}\int_0^T u(x,t)dt=0,\ \ x\in\mathbb{R}^n.$$

As above, we find

$$g(x)=-\frac{1}{NT}\int_0^T\ u(x,\eta)f_{1-\beta}(T-\eta)d\eta,\ \ x\in\mathbb{R}^n, \tag{17}$$

where u satisfies the equation

$$u(x,t)+\frac{1}{NT}\int_0^T\Big[u(x,\eta)*\int_0^t\ F_0(\tau)G_0(x,t-\tau)d\tau\Big]f_{1-\beta}(T-\eta)d\eta=0,\ \ (x,t)\in\bar{Q} \tag{18}$$

in the space $C\big([0,T];H^{s+\alpha+\gamma,p}(\mathbb{R}^n)\big)$ (here the symbol $*$ means the convolution in space variables).

This is a linear homogeneous integral equation of the form (15). We obtain the existence of such positive T that ensures the unique solvability of the equation (18) in the space $C\big([0,T];H^{s+\alpha+\gamma,p}(\mathbb{R}^n)\big)$, and therefore, $u=0$. Then from (17) we get $g(x)=0$, $x\in\mathbb{R}^n$. □

REFERENCES

1. V.V. Anh, N.N. Leonenko, *Spectral analysis of fractional kinetic equations with random datas*, J. of Statistical Physics, **104** (2001), №5/6, 1349–1387.
2. M. Caputo, *Linear model of dissipation whose Q is almost friequency independent, II*, Geophysical. Journal International, **13** (1967), №5, 529–539.
3. R.S. Damor, S. Kumar, A.K. Shukla, *Solution of fractional bioheat equation in terms of Fox's H-funtion*, Damor et al. Springer Plus, **5** (2016), 111. DOI 10.1186/s40064-016-1743-2
4. Duan Jun Sheng, *Time- and space-fractional partial differential equations*, J. Math. Phis., **46** (2005), 013504.
5. S.D. Eidelman, S.D. Ivasyshen, A.N. Kochubei, Analytic methods in the theory of differential and pseudo-differential equations of parabolic type, Birkhauser Verlag, Basel-Boston-Berlin, 2004.
6. R. Gorenflo, A. Iskenderov, Yu. Luchko, *Mapping between solutions of fractional diffusion-wave equations*, Fract. Calc. Appl. Anal., **3** (2000), 75–86.
7. A. Hanyga, *Multi-dimensional solutions for space-time-fractional diffusion equations*, Proc. R. Soc. London, **458** (2002), 429–450.
8. R. Hilfer (Ed.), Applications of Fractional Calculas in Physics, World Scientific, Singapore, 2000.
9. F. Huang, B. Guo, *General solutions to a class of time fractional partial differential equations*, Appl. Math. Mech., **31** (2010), 815–626.
10. M. Kirane, A. Lopushansky, H. Lopushanska, *Inverse problem for a time fractional differential equation with a time- and space-integral conditions*, Math. Meth. Appl. Sci., **46** (2023), №15, 16381–16393. DOI10.1002/mma.9453. https://doi.org/10.1002/mma.9453
11. M. Kirane, A. Lopushansky, H. Lopushanska, *Determination of two unknown functions of different variables in a time-fractional differential equation*, Math. Meth. Appl. Sci., (2024), 1–10. DOI 10.1002/mma.10539.
12. A.N. Kochubei, *Fractional-parabolic systems*, Potential Anal., **37** (2012), 1–30. https://doi.org/10.1007/s11118-011-9243-z
13. A. Lopushansky, *The Cauchy problem for an equation with fractional derivatives in Bessel potential spaces*, Sib. Math. J., **55** (2014), №6, 1089–1097. DOI:10.1134/30037446614060111
14. A. Lopushansky, O. Lopushansky, A. Szpila, *Fractional abstract Cauchy problem on complex interpolation scales*, FCAA, **23** (2020), №4, 1125–1140. DOI:10.1515/fca-2020-0057
15. Yu. Luchko, M. Yamamoto, *General time-fractional diffusion equation: some uniqueness and existence results for the initial-boundary-value problems*, Fract. Calc. Appl. Anal., **19** (2016), 675–695.

16. F. Mainardi, *The fundamental solutions for the fractional diffusion-wave equation*, Appl. Math. Lett., **9** (1996), №6, 23–28.
17. H.P. Lopushanska, A.O. Lopushanskyj, *Space-time fractional Cauchy problem in spaces of generalized functions*, Ukrainian Math. J., **64** (2013), №8, 1215–1230. doi: 10.1007/s11253-013-0711-z (translaited from Ukr. Mat. Zhurn., **64** (2012), №8, 1067–1080).
18. H. Lopushanska, A. Lopushansky, *Inverse problem with a time-integral condition for a fractional diffusion equation*, Math. Meth. Appl. Sci., **42** (2019), №6, 3327–3340. https://doi.org/10.1002/mma.5587
19. H. Lopushanska, A. Lopushansky, *Inverse problems for a time fractional diffusion equation in the Schwartz-type distributions*, Math. Meth. Appl. Sci., **44** (2021), №3, 2381–2392. https://doi.org/10.1002/mma
20. A. Lopushansky, O. Lopushansky, S. Sharyn, *Nonlinear inverse problem of control parameter determination for a space-time fractional diffusion equation*, Appl. Math. Comput., **390** (2021), 125589. https://doi.org/10.1016/j.amc.2020.125589
21. G. Da Prato, P. Grisvard, *Equations d'evolution abstraites non lineaires de type parabolique*, Ann. Mat. Pura. Appl. IV, **120** (1979), 329–396.
22. R.L. Magin, *Fractional calculas in Bioengineering: Part 1, Part 2, Part 3*, Critical Rewiews in Biomedical Engineering, **32** (2004), 1–104, 105–193, 195–377.
23. Krein S.G. Functional analysis (under the general editorship of Crane S.G.), Series "Reference Mathematical Library", M., 1972.

Lviv Polytechnic National University
Lviv, Ukraine
boyko.petro@gmail.com

Ivan Franko National University of Lviv
Lviv, Ukraine
lhp@ukr.net

Received 05.02.2025
Revised 23.08.2025