UDK 519.21

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CONTROL PROBLEM FOR THE MARKOV-MODULATED POISSON PROCESS IN THE DIFFUSION SCHEMA

S. A. Semenyuk, Ya. M. Chabanyuk, R. A. Chypurko, A. A. Lytvyn. *Control problem for the Markov-modulated Poisson process in the diffusion schema*, Mat. Stud. **64** (2025), 99–106.

This paper addresses the optimal control problem for a stochastic evolution system perturbed by a Markov-modulated Poisson process within a diffusion approximation framework. The considered system captures complex dynamics involving continuous evolution and discrete, state-dependent jumps, enabling the modeling of systems with regime-switching behavior or infrequent but significant events. The control function is constructed by minimizing a quality criterion through a stochastic optimization procedure. To analyze the asymptotic behavior of the system as the perturbation parameter vanishes, we derive the generator of the process and solve a corresponding singular perturbation problem. This allows us to prove the weak convergence of the stochastic system to a diffusion process. Furthermore, we establish sufficient conditions under which the control strategy converges almost surely to an optimal control. The obtained result makes it possible to study the rate of convergence of evolution under the optimal control for problems with a Markov-modulated Poisson perturbation.

Introduction. Stochastic evolutionary systems have gained significant attention recently due to their applicability across various scientific and engineering disciplines, including cybersecurity, financial modeling ([1]), and reliability theory. In [2], authors considered the optimal control problem for the case when an impulsive process introduces the perturbation. This work investigates an optimal control problem for a stochastic evolution system influenced by Markov-modulated Poisson (MMP, [3]) perturbations within a diffusion approximation framework. This type of perturbation arises from combining a Poisson process and a modulating Markov process that controls the intensity of jumps. The control function is defined via an extremum condition of a quality criterion, and the optimization procedure is designed to converge to the optimal solution almost surely under appropriate conditions. We find the form of the limit generators for the perturbed processes and the control of the dynamical system in the diffusion approximation schema and the stochastic optimization. Based on this, we prove weak convergence results and establish sufficient conditions for the convergence of the control function.

Problem statement. We consider the optimal control problem for a stochastic evolution system influenced by the Markov-modulated Poisson perturbations. The system dynamics are defined by a stochastic differential equation in which both the control function and the stochastic disturbances are subject to the influence of an underlying ergodic Markov process.

2020 *Mathematics Subject Classification:* 60H30, 49K45.
Keywords: random evolution; control problem; stochastic optimization procedure; Markov modulated Poisson process; diffusion approximation.
doi:10.30970/ms.64.1.99-106

The task is to determine a control strategy $u^\varepsilon(t)$ that minimizes quality criterion $J(y,u,x)$. To achieve this, we consider the diffusion schema approach, which allows us to approximate the behavior of the original perturbed system by an averaged deterministic counterpart in the limit. The aim is to analyze the asymptotic behavior of the coupled stochastic system and optimization procedure, establish the weak convergence of the controlled process, and derive sufficient conditions under which the control function converges to an optimal solution with probability one.

Let us consider an uniformly ergodic Markov process $x(t), t \geq 0$ in the standard phase space $(X, \mathbb{X})$. Here X is a complete, separable metric space,(that is, a Polish space), and $\mathbb{X}$ is its Borel σ-algebra of subsets of X. This process is defined by the generator ([7])

$$\mathbf{Q}\varphi(x) = q(x)\int_X P(x,dy)\left[\varphi(y)-\varphi(x)\right], \varphi \in \mathbb{B}(X), \tag{1}$$

here $\mathbb{B}(X)$ is a Banach space of bounded functions with supremum-norm

$$\|\varphi\| = \sup_{x\in X}|\varphi(x)|$$

([7, p.2]). Stochastic kernel $P(x,B), x \in X, B \in \mathbb{X}$ defines uniformly ergodic embedded Markov chain $x_i = x(\tau_i)$, $i \geq 0$, with stationary distribution $\rho(B), B \in \mathbb{X}$. Here $\tau_i, i \geq 0$ are the jump times of the process $x(t)$, so $0 = \tau_0 \leq \tau_1 \leq \ldots \leq \tau_i \leq \ldots$ Stationary distribution $\pi(B), B \in \mathbb{X}$ of the Markov process $x(t), t \geq 0$, is defined by the representation

$$\pi(dx)q(x) = q\rho(dx),\ q = \int_X \pi(dx)q(x).$$

Let us denote by $\mathbf{R_0}$ the potential operator of the generator $\mathbf{Q}$, which is defined by: $\mathbf{R_0} = \mathbf{\Pi} - (\mathbf{\Pi}+\mathbf{Q})^{-1}$, here $\mathbf{\Pi}\varphi(x) = \int_X \pi(dy)\varphi(y)$ is projector on the zeros subspace $Z_Q = \{\varphi\colon \mathbf{Q}\varphi \equiv 0\}$ of the operator $\mathbf{Q}$.

We investigate the stochastic evolution system in an ergodic Markovian environment given by the evolution equation ([4, p.54])

$$dy^\varepsilon(t) = C(y^\varepsilon(t), x(t/\varepsilon^2))dt + \varepsilon^{-1}a(t,u^\varepsilon(t))\tilde{N}_x(t)dt, \tag{2}$$

where $y^\varepsilon(t) \in D(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ is Skorokhod space of cadlag functions ([6, p.123], right-continuous with left limits) on $\mathbb{R}_{\geq 0}$ with values in $\mathbb{R}^n$. $a(t,u) \in C(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ represents the size and effect of the jump at time t and $u^\varepsilon(t) \in C(\mathbb{R}_{\geq 0})$ is a control function. And the function $C(y,x)$ with values in $\mathbb{R}^n$ and defined on $\mathbb{R}^n \times X$ describes the drift of the evolution process. Let also $C(y,\cdot) \in C(\mathbb{R}^n)$.

Asymptotic properties for such stochastic evolution system are investigated in [5].

Let $N_x^\varepsilon(t)$ be Poisson process counting the number of events that have occurred up to time t. The arrival rate of events at the moment t $\lambda(x(t/\varepsilon^2))$ is modulated by the Markov process of the state. That is the function $\lambda\colon X \to \mathbb{R}_{>0}$ defines (modulates) the jump rate of the associated Poisson process based on the current state $x \in X$ of the Markov process.

Define cumulative intensity (compensator) of Poisson process jumps up to time t by $\Lambda(t) = \int_0^t \lambda(x(s/\varepsilon^2))ds$.

Let us introduce a compensated Poisson process $\tilde{N}_x(t)$ defined as

$$\tilde{N}_x(t) = N_x^\varepsilon(t) - \Lambda(t) = N_x^\varepsilon(t) - \int_0^t \lambda(x(s/\varepsilon^2))ds. \tag{3}$$

Such a process is used to center the jumps, i.e., $E[\tilde{N}_x(t)] = 0$.

The generator for the process $\tilde{N}_x(t)$ (3) has the form ([8, p.104])

$$\tilde{\mathbf{W}}\varphi(N) = \lambda(x)(\mathbf{R}_+ - \mathbf{I})\varphi(N) - \lambda(x)\varphi'(N), \tag{4}$$

where $\mathbf{R}_+\varphi(N) = \varphi(N+1)$ and $\mathbf{I}\varphi(N) = \varphi(N)$. The generator $\tilde{\mathbf{W}}\varphi(N)$ is defined on the space of functions φ that are continuously differentiable with respect to the real-valued argument N, that is $\mathcal{D}(\tilde{\mathbf{W}}) = \{\varphi \in C^1(\mathbb{R})\}$.

Also, it can be proven that the averaging of the process $\tilde{N}_x(t)$ along the stationary distribution of the moduling Markov process equals zero [5, p. 445], i.e.

$$\mathbf{\Pi}\tilde{N}_x(t) = \int_X \pi(dx)\tilde{N}_x(t) = 0. \tag{5}$$

Stochastic optimization procedure for the control function. Let us introduce a quality criterion $J(y, u, x)$ that has a unique extremum (we will consider infinum for the sake of simplicity) for each value of process y and for each state x of the Markov process $x(t)$.

Such kind of quality criterion defines the control function [9] $u^\varepsilon(t)$ for equation (2) in the way

$$u^* = \arg\inf_u J(y, u, x).$$

Let also $J(\cdot, u, \cdot) \in C^2(\mathbb{R}^n)$, i.e. twice continuously differentiable. Then the control $u(t)$ is completely determined by the system

$$\frac{\partial J(y, u, x)}{\partial u_i} = 0, \quad i = \overline{1, n}.$$

Consider stochastic optimization procedure for control function $u^\varepsilon(t)$ in the form

$$du^\varepsilon(t) = \alpha(t)\nabla_{\beta(t)} J(y^\varepsilon(t), u^\varepsilon(t), x(t/\varepsilon^2))dt, \tag{6}$$

where $\alpha(t)$ is the learning rate and $\beta(t)$ is a difference scheme step.

$$\nabla_{\beta(t)} J(\cdot, u, \cdot) = \left\{ \frac{J(\cdot, u_i^+, \cdot) - J(\cdot, u_i^-, \cdot)}{2\beta(t)}, i = \overline{1, n} \right\}, \tag{7}$$
$$u_i^\pm = u \pm \beta(t)e_i, e_i = (0, \ldots, \underbrace{1}_{i\text{-th position}}, \ldots, 0), i = \overline{1, n}.$$

Functions $\alpha(t)$ and $\beta(t)$ are satisfying conditions $\alpha(t) \to 0$, $\beta(t) \to 0$ $(t \to 0)$.

Let general initial conditions have the form

$$y(0) = y_0, u(0) = u_0, x(0) = x_0. \tag{8}$$

Lemma 1. *Generator of the four-component process $(y^\varepsilon(t), u^\varepsilon(t), \tilde{N}_x(t), x(t/\varepsilon^2))$ has the asymptotical representation*

$$\mathbf{L}^\varepsilon(x)\varphi(y, u, w, x) = \varepsilon^{-2}\mathbf{Q}\varphi(y, u, w, x) + \varepsilon^{-1}\mathbf{A_t}\varphi(y, u, w, x) + \mathbf{B_t}(x)\varphi(y, u, w, x) +$$
$$+ \mathbf{C}(y, x)\varphi(y, u, w, x) + \mathbf{W}(x)\varphi(y, u, w, x), \tag{9}$$

where

$$\mathbf{A_t}\varphi(y, u, w, x) = a(t, u)w\varphi'_y(y, u, w, x),$$

$$\mathbf{B_t}(x)\varphi(y,u,w,x) = \alpha(t)\nabla_{\beta(t)}J(y,u,x)\varphi'_u(y,u,w,x), \tag{10}$$
$$\mathbf{C}(y,x)\varphi(y,u,w,x) = C(y,x)\varphi'_y(y,u,w,x),$$
$$\mathbf{W}(x)\varphi(y,u,w,x) = \lambda(x)(\mathbf{R}_+ - \mathbf{I})\varphi(y,u,w,x) - \lambda(x)\varphi'_w(y,u,w,x).$$

The generator $\mathbf{L}^\varepsilon(x)$ *is defined on the space of test functions defined on the space of test functions* $\varphi(y,u,w,x)$ *that are continuously differentiable in* y,u *and* w *and measurable and bounded in* x*, that is* $\mathcal{D}(\mathbf{L}^\varepsilon(x)) = \{\varphi \in C^1(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}) \otimes \mathbb{B}(X)\}$. *Recall here* $\mathbb{B}(X)$ *is a Banach space of bounded measurable functions on the Markov phase space* X.

Proof. Similar to lemma 1 in [8, p.103] let us consider the conditional expectation

$$E\Big(\varphi(y^\varepsilon(t+\Delta), u^\varepsilon(t+\Delta), \tilde{N}^\varepsilon(t+\Delta), x((t+\Delta)/\varepsilon^2)) - \varphi(y^\varepsilon(t), u^\varepsilon(t), \tilde{N}_x(t), x(t/\varepsilon^2))|$$
$$y^\varepsilon(t) = y, u^\varepsilon(t) = u, \tilde{N}_x(t) = w, x(t/\varepsilon^2) = x\Big) =$$
$$= E\left(\varphi(y_{t+\Delta}, u_{t+\Delta}, w_{t+\Delta}, x_{t+\Delta}) - \varphi(y,u,w,x)\right) =$$
$$= E\left(\varphi(y_{t+\Delta}, u_{t+\Delta}, w_{t+\Delta}, x_{t+\Delta}) - \varphi(y, u_{t+\Delta}, w_{t+\Delta}, x_{t+\Delta})\right) +$$
$$+E\left(\varphi(y, u_{t+\Delta}, w_{t+\Delta}, x_{t+\Delta}) - \varphi(y,u, w_{t+\Delta}, x_{t+\Delta})\right) + E\left(\varphi(y,u, w_{t+\Delta}, x_{t+\Delta}) - \varphi(y,u,w,x)\right).$$

According to (2): $y(t+\Delta) = y + C(y,x)\Delta + \varepsilon^{-1}a(t,u)w\Delta + o(\Delta)$, therefore

$$\lim_{\Delta\to 0}\frac{1}{\Delta}E(\varphi(y_{t+\Delta}, u_{t+\Delta}, w_{t+\Delta}, x_{t+\Delta}) - \varphi(y, u_{t+\Delta}, w_{t+\Delta}, x_{t+\Delta})) =$$
$$= \lim_{\Delta\to 0}\frac{1}{\Delta}E\left(\varphi(y + C(y,x)\Delta + \varepsilon^{-1}a(t,u)w\Delta + o(\Delta)), w_{t+\Delta}, x_{t+\Delta}) - \varphi(y, u_{t+\Delta}, w_{t+\Delta}, x_{t+\Delta})\right) =$$
$$= \lim_{\Delta\to 0}\frac{1}{\Delta}E\left(\varphi'_y(y, u_{t+\Delta}, w_{t+\Delta}, x_{t+\Delta})(C(y,x)\Delta + \varepsilon^{-1}a(t,u)w\Delta + o(\Delta))\right) =$$
$$= \left(C(y,x) + \varepsilon^{-1}a(t,u)w\right)\varphi'_y(y,u,w,x).$$

Using (6) we can write $u(t+\Delta) = u + \alpha(t)\nabla_{\beta(t)}J(y,u,x)\Delta + o(\Delta)$, therefore

$$\lim_{\Delta\to 0}\frac{1}{\Delta}E(\varphi(y, u_{t+\Delta}, w_{t+\Delta}, x_{t+\Delta}) - \varphi(y,u,w,x)) = \alpha(t)\nabla_{\beta(t)}J(y,u,x)\varphi'_u(y,u,w,x).$$

And the last term would according to (1) and (4) be written in the form

$$\lim_{\Delta\to 0}\frac{1}{\Delta}E(\varphi(y,u,w_{t+\Delta}, x_{t+\Delta}) - \varphi(y,u,w,x)) = \varepsilon^{-2}\mathbf{Q}\varphi(y,u,w,x) +$$
$$+\lambda(x)(\mathbf{R}_+ - \mathbf{I})\varphi(y,u,w,x) + \lambda(x)\varphi'_w(y,u,w,x).$$

Combining these results, we receive the lemma statement (9). □

Lemma 2. *Singular perturbation problem for the operator* (9) *on the test functions*

$$\varphi^\varepsilon(y,u,w,x) = \varphi(y,u,w) + \varepsilon\varphi_1(y,u,w,x) + \varepsilon^2\varphi_2(y,u,w,x),$$

has the solution in the form

$$\mathbf{L}^\varepsilon(x)\varphi^\varepsilon(y,u,w,x) = \mathbf{L}\varphi(y,u,w) + \varepsilon\theta(x)\varphi(y,u,w,x), \tag{11}$$

where the remaining term $\theta(x)$ *is uniformly bounded on* x.

Limit operator $\mathbf{L}$ *is defined by*

$$\mathbf{L\Pi} = \mathbf{\Pi B_t \Pi} + \mathbf{\Pi C}(u,x)\mathbf{\Pi} + \mathbf{\Pi W}(x)\mathbf{\Pi} + \mathbf{\Pi A_t R_0 A_t \Pi}. \tag{12}$$

Proof. Let us conduct the similar terms with respect to ε to proof the equality (11)

$$\mathbf{L}^\varepsilon(x)\varphi^\varepsilon(y,u,w,x) = \varepsilon^{-2}\mathbf{Q}\varphi(y,u,w) + \varepsilon^{-1}\left(\mathbf{Q}\varphi_1(y,u,w,x) + \mathbf{A_t}\varphi(y,u,w)\right) + \\ +\mathbf{Q}\varphi_2(y,u,w,x) + \mathbf{A_t}\varphi_1(y,u,w,x) + \left(\mathbf{B_t}(x) + \mathbf{C}(y,x) + \mathbf{W}(x)\right)\varphi(y,u,w) + \\ +\varepsilon\left(\mathbf{B_t}(x) + \mathbf{C}(y,x) + \mathbf{W}(x)\right)\varphi_1(u,w,x) + \varepsilon\mathbf{A_t}\varphi_2(u,w,x) + \\ +\varepsilon^2\left(\mathbf{B_t}(x) + \mathbf{C}(y,x) + \mathbf{W}(x)\right)\varphi_2(u,w,x).$$

Since $\varphi(y,u,w)$ does not depend on x, threrefore $\mathbf{Q}\varphi(y,u,w) = 0 \Leftrightarrow \varphi(y,u,w) \in Z_Q$. According to (5)

$$\mathbf{\Pi A_t}\varphi(y,u,w) = \mathbf{\Pi}a(t,u)w\varphi'_y(y,u,w) = a(t)\mathbf{\Pi}w\varphi'_y(y,u,w)|_{w=\tilde{N}_x(t)} = 0.$$

And this is a solution condition for the equation

$$\mathbf{Q}\varphi_1(y,u,w,x) + \mathbf{A_t}\varphi(y,u,w) = 0,$$

therefore we can calculate $\varphi_1(y,u,w,x)$ in the form

$$\varphi_1(y,u,w,x) = \mathbf{R_0 A_t}\varphi(y,u,w). \tag{13}$$

The next term would be written in the form

$$\mathbf{Q}\varphi_2(y,u,w,x) + \left(\mathbf{A_t R_0 A_t} + \mathbf{B_t}(x) + \mathbf{C}(y,x) + \mathbf{W}(x)\right)\varphi(y,u,w) = \mathbf{L}\varphi(y,u,w).$$

We can obtain limit process $\mathbf{L}$ in the form (12) using the solution condition of the previous equation. Then

$$\varphi_2(y,u,w,x) = \mathbf{R_0}\left(\mathbf{A_t R_0 A_t} + \mathbf{B_t}(x) + \mathbf{C}(y,x) + \mathbf{W}(x) - \mathbf{L}\right)\varphi(y,u,w),$$

and taking into account that $\mathbf{R_0 L} = 0$, we obtain

$$\varphi_2(y,u,w,x) = \mathbf{R_0}\left(\mathbf{A_t R_0 A_t} + \mathbf{B_t}(x) + \mathbf{C}(y,x) + \mathbf{W}(x)\right)\varphi(y,u,w). \tag{14}$$

Using (13) and (14) we can bring the neglectable term to the form $\varepsilon\theta(x)\varphi(y,u,w)$ and prove that $\theta(x)$ on the functions $\varphi(y,u,w)$ is bounded using the form of operators $\mathbf{A_t}$, $\mathbf{B_t}(x)$, $\mathbf{C}(y,x)$, $\mathbf{W}(x)$ and $\mathbf{R_0}$. □

Theorem 1. *The weak convergence holds true*

$$(y^\varepsilon(t), u^\varepsilon(t), N_x^\varepsilon(t)) \to (\hat{y}(t), \hat{u}(t), \hat{N}(t)), \quad \varepsilon \to 0.$$

Limit process $(\hat{y}(t), \hat{u}(t), \hat{N}(t))$ *is defined by the generator*

$$\mathbf{L}\varphi(y,u,w) = \hat{C}(y)\varphi'_y(y,u,w) + \frac{1}{2}\hat{A}(u)^2\varphi''_{yy}(y,u,w) + \hat{B}(y,u)\varphi'_u(y,u,w) + \\ +\frac{1}{2}\Lambda^2\varphi''_{ww}(y,u,w) + \mathbf{S}\varphi(y,u,w), \tag{15}$$

where

$$\hat{C}(y) = \mathbf{\Pi}C(y,x) = \int_X \pi(dx)C(y,x), \quad \hat{A}(u) = a(t,u)\sqrt{2\int_X \pi(dx)w\mathbf{R_0}w},$$

$$\hat{B}(y,u) = \alpha(t)\nabla_{\beta(t)}J(y,u), \quad J(y,u) = \mathbf{\Pi}J(y,u,x) = \int_X \pi(dx)J(y,u,x),$$

$$\Lambda = \sqrt{\mathbf{\Pi}\lambda(x)} = \sqrt{\int_X \pi(dx)\lambda(x)}, \quad \mathbf{S}\varphi(y,u,w) = \int_X \pi(dx)\lambda(x)\sum_{n=3}^{\infty}\frac{1}{n!}\varphi_w^{(n)}(y,u,w).$$

Proof. Let us perfrom Taylor decomposition of the operator $\mathbf{W}(x)$ taking into account the nature of the operator $\mathbf{R_0}$

$$(\mathbf{R_0}-\mathbf{I})\varphi(y,u,w)-\varphi'_w(y,u,w)=\varphi(y,u,w+1)-\varphi(y,u,w)-\varphi'_w(y,u,w)=$$
$$=\varphi(y,u,w)+\frac{\varphi'_w(y,u,w)}{1!}1+\frac{\varphi''_w(y,u,w)}{2!}1^2+\frac{\varphi'''_w(y,u,w)}{3!}1^3+\ldots-\varphi(y,u,w)-$$
$$-\varphi'_w(y,u,w)=\frac{1}{2}\varphi''_w(y,u,w)+\frac{1}{3!}\varphi'''_w(y,u,w)+\ldots=\frac{1}{2}\varphi''_w(y,u,w)+\sum_{n=3}^{\infty}\frac{1}{n!}\varphi_w^{(n)}(y,u,w).$$

Considering this and calculating the right part of the (12) we obtain the generator $\mathbf{L}$ in form (15). Theorem proof completion is made using the Theorem 3.2 from [7, p.79]. □

In this case, limit process $(\hat{y}(t),\hat{u}(t))$ for the control problem is defined by the stochastic differential equations:

$$d\hat{y}(t)=\hat{C}(\hat{y}(t))dt+\hat{A}(\hat{u}(t))dW(t),$$
$$d\hat{u}(t)=\alpha(t)\nabla_{\beta(t)}J(\hat{y}(t),\hat{u}(t))dt,$$

where $dW(t)$ is stochastic differential Ito of the Wiener process.

Theorem 2. *Let Lyapunov function $V(y,u)$ of the averaged system*

$$\frac{du}{dt}=\nabla_u J(y,u),$$

where $\nabla_u J(y,u)=\{\frac{\partial J}{\partial u_i}, i=\overline{1,n}\}$ for arbitrary value of the process y, exists and provides its exponential stability

$$\nabla_u J(y,u)V'_u(y,u)\leq -c_0V(y,u),\quad c_0>0. \tag{16}$$

Suppose that

$$|V'_u(y,u)|\leq c_1(1+V(y,u)),\quad c_1>0. \tag{17}$$

Let $\nabla_u J(y,u,x)$ is uniformly bounded on state space X, that is there exists $0<M(y,u)\leq c_2$ (where c_2 is some constant) such that

$$\sup_{x\in X}\|\nabla_u J(y,u,x)\|\leq M(y,u) \tag{18}$$

and

$$\int_{t_0}^{\infty}\alpha(t)dt=\infty,\int_{t_0}^{\infty}\alpha(t)\beta(t)dt<\infty, t_0>0.$$

Then the solution for control problem (2), (6), (8) *converges with probability 1 to the optimal control of the averaged system, i.e.*

$$P\Big\{\lim_{\varepsilon\to 0}u^\varepsilon(t)=\hat{u}(t)\Big\}=1.$$

Proof. Let us use Taylor expansion for each term with $u_i^{\pm}$ of the finite difference approximation (7), we have

$$J(y,u+\beta e_i,x)=J(y,u,x)+\beta\,\partial_{u_i}J(y,u,x)+\frac{\beta^2}{2}\partial^2_{u_iu_i}J(y,u^*,x),$$

$$J(y, u - \beta e_i, x) = J(y, u, x) - \beta\, \partial_{u_i} J(y, u, x) + \frac{\beta^2}{2} \partial^2_{u_i u_i} J(y, u^{**}, x),$$

where u^* and u^{**} lie between u and $u \pm \beta e_i$, respectively. Subtracting one from another, and dividing by $2\beta(t)$, we obtain $\left[\nabla_{\beta(t)} J(y, u, x)\right]_i = \partial_{u_i} J(y, u, x) + \mathcal{O}(\beta(t))$.

Therefore, there exists such constant c_3 that the finite-difference approximation satisfies

$$\nabla_{\beta(t)} J(y, u, x) = \nabla_u J(y, u, x) + \mathcal{O}(\beta(t)) \leq \nabla_u J(y, u, x) + c_3 \beta(t). \tag{19}$$

Using the uniform boundness of $\nabla_u J(y, u, x)$ (18) it can be shown that

$$\nabla_u J(y, u, x) - \nabla_u J(y, u) \leq 2c_2. \tag{20}$$

Now we apply the generator of the control function $\mathbf{B_t}(x)$ (10) to the Lyapunov function $V(y, u)$

$$\mathbf{B_t}(x) V(y, u) = \alpha(t) \nabla_{\beta(t)} J(y, u, x) V'_u(y, u).$$

Using the results (19) and (20) together with Theorem conditions (16)–(17) it can be rewritten

$$\begin{gathered}
\alpha(t) \nabla_{\beta(t)} J(y, u, x) V'_u(y, u) \leq \alpha(t) \nabla_u J(y, u, x) V'_u(y, u) + c_3 \alpha(t) \beta(t) V'_u(y, u) \pm \\
\pm \alpha(t) \nabla_u J(y, u) V'_u(y, u) \leq \\
\leq \alpha(t)(\nabla_u J(y, u, x) - \nabla_u J(y, u)) V'_u(y, u) + \alpha(t) \nabla_u J(y, u) V'_u(y, u) + c_3 \alpha(t) \beta(t) V'_u(y, u) \leq \\
\leq 2\alpha(t) c_3 V'_u(y, u) - c_0 \alpha(t) V(y, u) + \alpha(t) \beta(t) V'_u(y, u) \leq \\
\leq -c_0 \alpha(t) V(y, u) + c_1(2c_3 \alpha(t) + \alpha(t)\beta(t))(1 + V(y, u)).
\end{gathered}$$

Thus $\mathbf{B_t}(x) V(y, u) \leq -c_0 \alpha(t) V(y, u) + c_1(2c_3 \alpha(t) + \alpha(t)\beta(t))(1 + V(y, u))$. Then the theorem proof completion is made similar to the Theorem 2 in [2, p.111]. □

Conclusions. This paper studied the optimal control problem for a stochastic evolution system perturbed by a Markov-modulated Poisson process within a diffusion approximation framework. By formulating the system as a singularly perturbed stochastic differential equation, we managed to derive the generator of the evolution process and analyze its asymptotic behavior. The obtained result allows us to construct the limit process and the stochastic optimization procedure for the initial control problem for stochastic evolution with the Markov-modulated Poisson perturbations and control, determined by the condition for the extremum of the quality criterion function. Sufficient conditions for the convergence of the procedure to optimal control with probability one are obtained for the diffusion approximation schema.

Further work may include investigation of the fluctuations of the evolution system ([10, p.717]) while optimal control would be applied. Also, we can consider such perturbed evolution systems in the non-traditional approximation schemas like Poisson or Levy ([2]).

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Received 15.12.2024
Revised 12.09.2025