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UNIQUENESS OF DIFFERENTIAL POLYNOMIAL OF MEROMORPHIC FUNCTION WITH ITS q-SHIFT

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In the paper, we apply the concept of weighted sharing to study the uniqueness problems of differential polynomial of meromorphic function of zero order with its q-shift. The results of the paper improve and extend some recent results due to H. P. Waghamore and M. M. Manakame [Int. J. Open Problems Compt. Math., 18 (2025), 22-34].

A typical theorem obtained in the paper is as follows:

Let P be a polynomial, $f(z)$ be a non-constant meromorphic function of zero-order. Suppose that q is a non-zero complex constant, $\eta \in \mathbb{C}$ and n is an integer satisfying $n \geq m+3\tau+3\Omega+6$, where $m=\deg P$, $\tau=\sum_{j=1}^{s}\mu_j$ and $\Omega=\sum_{j=1}^{s} j\mu_j$. If $f^n(z)P(f(z))\prod_{j=1}^{s} f^{(j)}(z)^{\mu_j}$ and $f^n(qz+\eta)P(f(qz+\eta))\prod_{j=1}^{s} f^{(j)}(qz+\eta)^{\mu_j}$ share $(1,2)$ and (∞,∞), then

$$f^n(z)P(f(z))\prod_{j=1}^{s} f^{(j)}(z)^{\mu_j} \equiv f^n(qz+\eta)P(f(qz+\eta))\prod_{j=1}^{s} f^{(j)}(qz+\eta)^{\mu_j}.$$

Three other similar theorems are also obtained in the paper.

1. Introduction, definitions and results. In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations of the Nevanlinna's theory of meromorphic functions as explained in [4], [8] and [19]. For a nonconstant meromorphic function $f(z)$, we denote by $T(r,f)$ the Nevanlinna characteristic function of $f(z)$ and by $S(r,f)$ any quantity satisfying $S(r,f)=o\{T(r,f)\}$ for all r outside a possible exceptional set of the finite logarithmic measure. We say that the meromorphic function $\alpha(z)$ is a small function of $f(z)$, if $T(r,\alpha)=S(r,f)$. We denote $S(f)$ by the family of all small functions including all constants and $\hat{S}=S(f)\cup\{\infty\}$.

Let k be a positive integer or infinity and $a\in\mathbb{C}\cup\{\infty\}$. Set $E(a,f)=\{z: f(z)-a=0\}$, where a zero with multiplicity k is counted k times. If the zeros are counted only once, then we denote the set by $\overline{E}(a,f)$. Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions. If $E(a,f)=E(a,g)$, then we say that $f(z)$ and $g(z)$ share the value a CM (counting multiplicities). If $\overline{E}(a,f)=\overline{E}(a,g)$, then we say that $f(z)$ and $g(z)$ share the value a IM (ignoring multiplicities). We denote by $E_{k)}(a,f)$ the set of all a-points of $f(z)$ with multiplicities not exceeding k, where an a-point is counted according to its multiplicity. Also we denote by $\overline{E}_{k)}(a,f)$ the set of all distinct a-points of $f(z)$ with multiplicities not exceeding k. Throughout the paper, we denote by

$$\rho(f)=\overline{\lim_{r\longrightarrow\infty}}\frac{\log T(r,f)}{\log r}$$

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the order of $f(z)$ (see [4], [8] and [19]). We define difference operators by

$$\Delta_\eta f(z) = f(z+\eta) - f(z), \quad \Delta_\eta^n f(z) = \Delta_\eta^{n-1}(\Delta_\eta f(z)),$$

where η is a nonzero complex number and $n \geq 2$ is a positive integer. If $\eta = 1$, we denote $\Delta_\eta f(z) = \Delta f(z)$. Analogously, we define q shift and q difference operators by

$$E_{q,\eta} f(z) = f(qz+\eta) \text{ and } \triangle_{q,\eta} f(z) = f(qz+\eta) - f(z),$$

where q is a nonzero complex number. In addition, we need the following definitions.

Let $f(z)$ be a nonconstant meromorphic function. An expression of the form

$$P[f] = \sum_{k=1}^{r} a_k(z) \prod_{j=0}^{s} f^{(j)}(z)^{l_{kj}}, \tag{1}$$

where $a_k(z) \in S(f)$ for $k \in \{1, 2, \ldots, r\}$ and l_{kj} are nonnegative integers for $k \in \{1, 2, \ldots, r\}$; $j \in \{0, 1, 2, \ldots, s\}$ and $d = \sum_{j=0}^{s} l_{kj}$ for each $k \in \{1, 2, \ldots, r\}$ is called ([9]) a *homogeneous differential polynomial of degree d generated by* $f(z)$. Also we denote the quantity $Q = \max_{1\leq k\leq r} \sum_{j=0}^{s} j l_{kj}$.

Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of $f(z)$ where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say ([7]) that $f(z)$ and $g(z)$ *share the value a with weight k*.

The definition implies that if $f(z)$ and $g(z)$ share a value a with weight k, then z_0 is an a-point of $f(z)$ with multiplicity $m(\leq k)$ if and only if it is an a-point of $g(z)$ with multiplicity $m(\leq k)$ and z_0 is an a-point of $f(z)$ with multiplicity $m(> k)$ if and only if it is an a-point of $g(z)$ with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write $f(z)$ and $g(z)$ share (a, k) to mean that $f(z)$ and $g(z)$ share the value a with weight k. Clearly if $f(z)$ and $g(z)$ share (a, k) then $f(z)$ and $g(z)$ share (a, p) for any integer p, $0 \leq p < k$. Also we note that $f(z)$ and $g(z)$ share a IM or CM if and only if $f(z)$ and $g(z)$ share $(a, 0)$ or (a, ∞) respectively.

Let $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid= 1)$ the counting function of simple a-points of $f(z)$. For a positive integer k we denote by $N(r, a; f \mid\leq k)$ the counting function of those a-points of $f(z)$ (counted with proper multiplicities) whose multiplicities are not greater than k. By $\overline{N}(r, a; f \mid\leq k)$ we denote the corresponding reduced counting function. Analogously, we can define $N(r, a; f \mid\geq k)$ and $\overline{N}(r, a; f \mid\geq k)$. Clearly, $\overline{N}(r, a; f) = N(r, a; f \mid= 1) + \overline{N}(r, a; f \mid\geq 2)$.

Let k be a positive integer or infinity. We denote by $N_k(r, a; f)$ the counting function of all a-points of $f(z)$, where an a-point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then $N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \ldots + \overline{N}(r, a; f \mid\geq k)$. Clearly, $N_1(r, a; f) = \overline{N}(r, a; f)$.

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions such that $f(z)$ and $g(z)$ share the value a IM and z_0 be an a-point of $f(z)$ with multiplicity p and an a-point of $g(z)$ with multiplicity q. We denote by $\overline{N}_L(r, a; f)$ the reduced counting function of those a-points of $f(z)$ for which $p > q$. In the same way, we can define $\overline{N}_L(r, a; g)$. Also we denote by $N_E^{1)}(r, a; f)$ the counting function of those a-points of $f(z)$ where $p = q = 1$.

Let $f(z)$ and $g(z)$ share the value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of $f(z)$ whose multiplicities differ from multiplicities of the corresponding a-points of $g(z)$. Clearly,

$$\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g).$$

Recently, some interest of mathematicians has been generated by studying q difference analogues in the value distribution and uniqueness theory of meromorphic and entire functions. A number of remarkable research works (see [5], [12], [13], [15], [16]) have been published.

For the q-shift polynomial of meromorphic function, J. Zhang and R. Korhonen [22] studied value distribution of q-shift polynomial of meromorphic function and obtained the following result.

Theorem A ([22]). *Let $f(z)$ be a transcendental meromorphic (resp. entire) function of zero order and $q \in \mathbb{C} \setminus \{0\}$ be a constant. Then for $n \geq 6$ (resp. $n \geq 2$), $f^n(z)f(qz)$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often.*

Example 1 ([22]). The zero-order growth restriction in Theorem A cannot be extended to finite order. This can be seen by taking $f(z) = e^z$ and $q = -n$. Then $f^n(z)f(qz) \equiv 1$.

In 2019, K. Meng and G. Liu [11] obtained a number of results for entire and meromorphic functions of zero order. Typical here are the following statements.

Theorem B ([11]). *Let $f(z)$ be a non-constant meromorphic function of zero-order. Suppose that q is a non-zero complex constant, $\eta \in \mathbb{C}$ and n is an integer satisfying $n \geq 7$. If $f^n(z)$ and $f^n(qz+\eta)$ share $(1,2)$, $f(z)$ and $f(qz+\eta)$ share (∞,∞), then $f(z) \equiv tf(qz+\eta)$, where t is a constant satisfying $t^n = 1$.*

Theorem C ([11]). *Let $f(z)$ be a non-constant meromorphic function of zero-order. Suppose that q is a non-zero complex constant, $\eta \in \mathbb{C}$ and n is an integer satisfying $n \geq 7$. If $E_{3)}(1, f^n(z)) = E_{3)}(1, f^n(qz+\eta))$, $f(z)$ and $f(qz+\eta)$ share (∞,∞), then $f(z) \equiv tf(qz+\eta)$, where t is a constant satisfying $t^n = 1$.*

In 2025, H.P. Waghamore and M.M. Manakame ([14]) considered a more general polynomial of the form $f^n(z)P(f(z))$ and its q shift $f^n(qz+\eta)P(f(qz+\eta))$, where $P(z) = a_m z^m + a_{m-1}z^{m-1} + \ldots + a_1 z + a_0$ ($a_j \in \mathbb{C}$, $a_m \neq 0$) is a polynomial of degree m, and obtained a number of theorems also in the class of meromorphic functions of zero order. Typical here is the following statement.

Theorem D ([14]). *Let $f(z)$ be a non-constant meromorphic function of zero-order. Suppose that q is a non-zero complex constant, $\eta \in \mathbb{C}$ and n is an integer satisfying $n \geq m+7$, where $m = \deg P$. If $f^n(z)P(f(z))$ and $f^n(qz+\eta)P(f(qz+\eta))$ share $(1,2)$ and $(\infty,0)$, then*

$$f^n(z)P(f(z)) \equiv f^n(qz+\eta)P(f(qz+\eta)).$$

Due to the questions of H.P. Waghamore and M.M. Manakame, we study the following differential polynomial of the form $f^n(z)P(f(z))\prod_{j=1}^{s} f^{(j)}(z)^{\mu_j}$ and its q-shift $f^n(qz+\eta)P(f(qz+\eta))\prod_{j=1}^{s} f^{(j)}(qz+\eta)^{\mu_j}$, where $\mu_j (j = 1,\ 2, \ldots, s)$ are nonnegative integers, which is the motivation of the present paper.

Theorem 1. *Let $f(z)$ be a non-constant meromorphic function of zero-order. Suppose that q is a non-zero complex constant, $\eta \in \mathbb{C}$ and n is an integer satisfying $n \geq m + 3\tau + 3\Omega + 6$, where $m = \deg P$, $\tau = \sum_{j=1}^{s} \mu_j$ and $\Omega = \sum_{j=1}^{s} j\mu_j$.*

If $f^n(z)P(f(z))\prod_{j=1}^{s} f^{(j)}(z)^{\mu_j}$ and $f^n(qz+\eta)P(f(qz+\eta))\prod_{j=1}^{s} f^{(j)}(qz+\eta)^{\mu_j}$ share $(1,2)$ and (∞,∞), then

$$f^n(z)P(f(z))\prod_{j=1}^{s} f^{(j)}(z)^{\mu_j} \equiv f^n(qz+\eta)P(f(qz+\eta))\prod_{j=1}^{s} f^{(j)}(qz+\eta)^{\mu_j}.$$

Theorem 2. *Let $f(z)$ be a non-constant meromorphic function of zero-order. Suppose that q is a non-zero complex constant, $\eta \in \mathbb{C}$ and n is an integer satisfying $n \geq m + 3\tau + 3\Omega + 7$, where $m = \deg P$, $\tau = \sum_{j=1}^{s} \mu_j$ and $\Omega = \sum_{j=1}^{s} j\mu_j$.*

If $f^n(z)P(f(z)) \prod_{j=1}^{s} f^{(j)}(z)^{\mu_j}$ and $f^n(qz+\eta)P(f(qz+\eta)) \prod_{j=1}^{s} f^{(j)}(qz+\eta)^{\mu_j}$ share $(1, 2)$ and $(\infty, 0)$, then

$$f^n(z)P(f(z)) \prod_{j=1}^{s} f^{(j)}(z)^{\mu_j} \equiv f^n(qz+\eta)P(f(qz+\eta)) \prod_{j=1}^{s} f^{(j)}(qz+\eta)^{\mu_j}.$$

Theorem 3. *Let $f(z)$ be a non-constant meromorphic function of zero-order. Suppose that q is a non-zero complex constant, $\eta \in \mathbb{C}$ and n is an integer satisfying $n \geq m + 3\tau + 3\Omega + 6$, where $m = \deg P$, $\tau = \sum_{j=1}^{s} \mu_j$ and $\Omega = \sum_{j=1}^{s} j\mu_j$.*

If $E_{3)}(1, f^n(z)P(f(z)) \prod_{j=1}^{s} f^{(j)}(z)^{\mu_j}) = E_{3)}(1, f^n(qz+\eta)P(f(qz+\eta)) \prod_{j=1}^{s} f^{(j)}(qz+\eta)^{\mu_j})$, $f^n(z)P(f(z)) \prod_{j=1}^{s} f^{(j)}(z)^{\mu_j}$ and $f^n(qz+\eta)P(f(qz+\eta)) \prod_{j=1}^{s} f^{(j)}(qz+\eta)^{\mu_j}$ share (∞, ∞), then

$$f^n(z)P(f(z)) \prod_{j=1}^{s} f^{(j)}(z)^{\mu_j} \equiv f^n(qz+\eta)P(f(qz+\eta)) \prod_{j=1}^{s} f^{(j)}(qz+\eta)^{\mu_j}.$$

Theorem 4. *Let $f(z)$ be a non-constant meromorphic function of zero-order. Suppose that q is a non-zero complex constant, $\eta \in \mathbb{C}$ and n is an integer satisfying $n \geq m + 3\tau + 3\Omega + 7$, where $m = \deg P$, $\tau = \sum_{j=1}^{s} \mu_j$ and $\Omega = \sum_{j=1}^{s} j\mu_j$.*

If $E_{3)}(1, f^n(z)P(f(z)) \prod_{j=1}^{s} f^{(j)}(z)^{\mu_j}) = E_{3)}(1, f^n(qz+\eta)P(f(qz+\eta)) \prod_{j=1}^{s} f^{(j)}(qz+\eta)^{\mu_j})$, $f^n(z)P(f(z)) \prod_{j=1}^{s} f^{(j)}(z)^{\mu_j}$ and $f^n(qz+\eta)P(f(qz+\eta)) \prod_{j=1}^{s} f^{(j)}(qz+\eta)^{\mu_j}$ share $(\infty, 0)$, then

$$f^n(z)P(f(z)) \prod_{j=1}^{s} f^{(j)}(z)^{\mu_j} \equiv f^n(qz+\eta)P(f(qz+\eta)) \prod_{j=1}^{s} f^{(j)}(qz+\eta)^{\mu_j}.$$

2. Lemmas. Below we state some lemmas which will be needed in the sequel. Denote: $H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right)$, where F and G are nonconstant meromorphic functions defined in the complex plane $\mathbb{C}$.

Lemma 1 ([9]). *Let $f(z)$ be a nonconstant meromorphic function and P[f] be defined by (1). Then $T(r, P[f]) \leq dT(r, f) + Q\overline{N}(r, \infty; f) + S(r, f)$ and*

$$N(r, 0; P[f]) \leq T(r, P[f]) - dT(r, f) + dN(r, 0; f) + S(r, f) \leq Q\overline{N}(r, \infty; f) + dN(r, 0; f) + S(r, f).$$

Lemma 2 ([17]). *Let $f(z)$ be a nonconstant meromorphic function and let $P(f) = a_m f^m + a_{m-1} f^{m-1} + \ldots + a_1 f + a_0$ be a polynomial of degree m, where $a_k \in S(f)$ for $k \in \{0, 1, \ldots, m\}$, $a_m \neq 0$. Then $T(r, P(f)) = mT(r, f) + S(r, f)$.*

Lemma 3 ([10]). *Let $f(z)$ be a nonconstant meromorphic function of zero order and q, η be two nonzero complex constants. Then we have $m\left(r, f(qz+\eta)/f(z)\right) = S(r, f)$.*

Lemma 4 ([4, 8, 19]). *Let $f(z)$ be a transcendental meromorphic function and k be a positive integer. Then $m\big(r, f^{(k)}(z)/f(z)\big) = S(r, f)$.*

Lemma 5 ([21]). *Let $f(z)$ be a transcendental meromorphic function of zero order and q, η be two nonzero complex constants. Then*

$$T(r, f(qz+\eta)) = T(r, f(z)) + S(r, f), \quad N(r, \infty; f(qz+\eta)) \leq N(r, \infty; f(z)) + S(r, f),$$
$$N(r, 0; f(qz+\eta)) \leq N(r, 0; f(z)) + S(r, f), \quad \overline{N}(r, \infty; f(qz+\eta)) \leq \overline{N}(r, \infty; f(z)) + S(r, f),$$
$$\overline{N}(r, 0; f(qz+\eta)) \leq \overline{N}(r, 0; f(z)) + S(r, f).$$

Lemma 6 ([1]). *Let F and G be two non-constant meromorphic functions. Suppose F and G share $(1,2)$ and (∞,k) where $0\le k\le\infty$. If $H\not\equiv 0$, then*

$$T(r,F)\le N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+ +\overline{N}_*(r,\infty;F,G)+S(r,F)+S(r,G).$$

Lemma 7 ([1]). *Let F and G be two non-constant meromorphic functions. Suppose $E_{3)}(1,F)=E_{3)}(1,G)$, F and G share (∞,k) where $0\le k\le\infty$. If $H\not\equiv 0$, then*

$$T(r,F)+T(r,G)\le 2N_2(r,0;F)+2N_2(r,0;G)+2\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)+ +2\overline{N}_*(r,\infty;F,G)+S(r,F)+S(r,G).$$

Lemma 8. *Let $f(z)$ be a transcendental meromorphic function of zero order and $\tau=\sum_{j=1}^{s}\mu_j$, $\Omega=\sum_{j=1}^{s}j\mu_j$. Then*

$$(n+m-\tau-\Omega)T(r,f)+S(r,f)\le T(r,f^n(z)P(f(z))\prod_{j=1}^{s}f^{(j)}(z)^{\mu_j})\le \le(n+m+\tau+\Omega)T(r,f)+S(r,f).$$

In addition, if $f(z)$ is a transcendental entire function of zero order, then

$$T(r,f^n(z)P(f(z))\prod_{j=1}^{s}f^{(j)}(z)^{\mu_j})=(n+m+\tau)T(r,f)+S(r,f).$$

Proof. Let $f(z)$ be transcendental entire function of zero order and

$$F=f^n(z)P(f(z))\prod_{j=1}^{s}f^{(j)}(z)^{\mu_j}.$$

Using Lemmas 2 and 4, we obtain

$$T(r,F)\le m(r,f^{n+\tau}(z)P(f(z)))+m\left(r,\prod_{j=1}^{s}\frac{f^{(j)}(z)^{\mu_j}}{f(z)^{\mu_j}}\right)+S(r,f)\le$$

$$\le(n+m+\tau)T(r,f)+S(r,f). \tag{2}$$

On the other hand,

$$(n+m+\tau)T(r,f)=T(r,f^{n+\tau}(z)P(f(z)))+S(r,f)\le m\left(r,\prod_{j=1}^{s}\frac{f(z)^{\mu_j}}{f^{(j)}(z)^{\mu_j}}\right)+ +m\left(r,f^{n+\tau}(z)P(f(z))\prod_{j=1}^{s}\frac{f^{(j)}(z)^{\mu_j}}{f(z)^{\mu_j}}\right)+S(r,f)\le T(r,F)+S(r,f). \tag{3}$$

From (2) and (3), we have

$$T(r,F)=T(r,f^n(z)P(f(z))\prod_{j=1}^{s}f^{(j)}(z)^{\mu_j})=(n+m+\tau)T(r,f)+S(r,f).$$

If $f(z)$ is transcendental meromorphic function of zero order, using Lemmas 1 and 2 we get

$$T(r,F)\le T(r,f^n(z)P(f(z)))+T\left(r,\prod_{j=1}^{s}f^{(j)}(z)^{\mu_j}\right)+S(r,f)\le(n+m)T(r,f)+\tau T(r,f)+$$

$$\Omega\overline{N}(r,\infty;f(z))+S(r,f)\le(n+m+\tau+\Omega)T(r,f)+S(r,f). \tag{4}$$

On the other hand

$$(n+m+\tau)T(r,f)=T(r,f^{n+\tau}(z)P(f(z)))+S(r,f)\le T\left(r,\prod_{j=1}^{s}\frac{f(z)^{\mu_j}}{f^{(j)}(z)^{\mu_j}}\right)+ +T\left(r,f^{n+\tau}(z)P(f(z))\prod_{j=1}^{s}\frac{f^{(j)}(z)^{\mu_j}}{f(z)^{\mu_j}}\right)+S(r,f)\le T(r,F)+(2\tau+\Omega)T(r,f)+S(r,f). \tag{5}$$

From (4) and (5), we get

$$(n+m-\tau-\Omega)T(r,f)+S(r,f)\le T(r,f^n(z)P(f(z))\prod_{j=1}^{s}f^{(j)}(z)^{\mu_j})\le \le(n+m+\tau+\Omega)T(r,f)+S(r,f).$$

□

Using Lemmas 3, 5 and similar arguments as in Lemma 8 we get the following Lemma.

Lemma 9. *Let $f(z)$ be a transcendental meromorphic function of zero order and $\tau = \sum_{j=1}^{s} \mu_j$, $\Omega = \sum_{j=1}^{s} j\mu_j$. Then*

$$(n+m-\tau-\Omega)T(r,f) + S(r,f) \le T(r, f^n(qz+\eta)P(f(qz+\eta))\prod_{j=1}^{s} f^{(j)}(qz+\eta)^{\mu_j}) \le$$
$$\le (n+m+\tau+\Omega)T(r,f) + S(r,f).$$

In addition, if $f(z)$ is a transcendental entire function of zero order, then

$$T(r, f^n(qz+\eta)P(f(qz+\eta))\prod_{j=1}^{s} f^{(j)}(qz+\eta)^{\mu_j}) = (n+m+\tau)T(r,f) + S(r,f).$$

3. Proof of the Theorems.

Proof of Theorem 1. Let

$F = f^n(z)P(f(z))\prod_{j=1}^{s} f^{(j)}(z)^{\mu_j}$ and $G = f^n(qz+\eta)P(f(qz+\eta))\prod_{j=1}^{s} f^{(j)}(qz+\eta)^{\mu_j}$.

The functions F and G are nonconstant meromorphic functions that share $(1,2)$ and (∞,∞). If possible, we may assume that $H \not\equiv 0$. Then we obtain from Lemma 6

$$T(r,F) \le N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,\infty;F) +$$
$$+\overline{N}(r,\infty;G) + \overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G). \tag{6}$$

As F and G share (∞,∞), it is obvious that $\overline{N}_*(r,\infty;F,G) = S(r,F)$. Therefore, using Lemmas 1, 2, 5, 8, 9 we get from (6)

$$(n+m-\tau-\Omega)T(r,f) \le 2\overline{N}(r,0;f(z)) + N(r,0;P(f(z))) + N(r,0;\prod_{j=1}^{s} f^{(j)}(z)^{\mu_j}) +$$
$$+2\overline{N}(r,0;f(qz+\eta)) + N(r,0;P(f(qz+\eta))) + N(r,0;\prod_{j=1}^{s} f^{(j)}(qz+\eta)^{\mu_j}) +$$
$$+2\overline{N}(r,\infty;f(z)) + S(r,f) \le (2m+2\tau+2\Omega+6)T(r,f) + S(r,f), \tag{7}$$

i.e., $(n-m-3\tau-3\Omega-6)T(r,f) \le S(r,f)$, which contradicts that $n \ge m+3\tau+3\Omega+6$. Therefore we must have $H \equiv 0$. Then $\frac{F''}{F'} - \frac{2F'}{F-1} = \frac{G''}{G'} - \frac{2G'}{G-1}$. Integrating both side twice we get from above $\frac{1}{F-1} = \frac{A}{G-1} + B$, where $A(\ne 0)$ and B are constants. So,

$$G = \frac{(B-A)F + (A-B-1)}{BF-(B+1)} \tag{8}$$

We now discuss the following three cases separately.

Case 1. Suppose that $B \ne 0, -1$. Then from (8) we have $\overline{N}(r,\frac{B+1}{B};F) = \overline{N}(r,\infty;G)$. Using Lemmas 1, 2, 5, 8 we obtain from second fundamental theorem of Nevanlinna,

$$(n+m-\tau-\Omega)T(r,f) \le T(r,F) + S(r,f) \le \overline{N}(r,0;F) + \overline{N}\Big(r,\frac{B+1}{B};F\Big) + \overline{N}(r,\infty;F) +$$
$$+S(r,f) \le \overline{N}(r,0;F) + \overline{N}(r,\infty;G) + \overline{N}(r,\infty;F) + S(r,f) \le \overline{N}(r,0;f(z)) + N(r,0;P(f(z))) +$$
$$+N(r,0;\prod_{j=1}^{s} f^{(j)}(z)^{\mu_j}) + 2\overline{N}(r,\infty;f(z)) + S(r,f) \le (m+\tau+\Omega+3)T(r,f) + S(r,f), \tag{9}$$

i.e., $(n-2\tau-2\Omega-3)T(r,f) \le S(r,f)$, which contradicts that $n \ge m+3\tau+3\Omega+6$.

Case 2. Suppose that $B = -1$. Then from (8) we have

$$G = \frac{(A+1)F - A}{F}. \tag{10}$$

Subcase 1. Suppose $A \ne -1$. Then from (10) we have $\overline{N}\big(r,\frac{A}{A+1};F\big) = \overline{N}(r,0;G)$. Using Lemmas 1, 2, 5, 8, 9 we obtain from second fundamental theorem of Nevanlinna,

$$(n+m-\tau-\Omega)T(r,f) \le T(r,F)+S(r,f) \le \overline{N}(r,0;F)+\overline{N}\Big(r,\frac{A}{A+1};F\Big)+$$
$$+\overline{N}(r,\infty;F)+S(r,f) \le \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+S(r,f) \le$$
$$\le \overline{N}(r,0;f(z))+N(r,0;P(f(z))+N(r,0;\prod_{j=1}^{s} f^{(j)}(z)^{\mu_j})+\overline{N}(r,0;f(qz+\eta))+$$
$$+N(r,0;P(f(qz+\eta))+N(r,0;\prod_{j=1}^{s} f^{(j)}(qz+\eta)^{\mu_j})+\overline{N}(r,\infty;f(z))+S(r,f) \le$$
$$\le (2m+2\tau+2\Omega+3)T(r,f)+S(r,f), \tag{11}$$

i.e., $(n-m-3\tau-3\Omega-3)T(r,f) \le S(r,f)$, which contradicts that $n \ge m+3\tau+3\Omega+6$.

Subcase 2. Let $A=-1$. Then from (10) we get $FG=1$ i.e.,

$$f^n(z)P(f(z))\prod_{j=1}^{s} f^{(j)}(z)^{\mu_j} f^n(qz+\eta)P(f(qz+\eta))\prod_{j=1}^{s} f^{(j)}(qz+\eta)^{\mu_j} \equiv 1.$$

From above it is clear that $f(z)$ can't have any zeros or poles. Therefore, $\overline{N}(r,0;f(z)) = S(r,f) = \overline{N}(r,\infty;f(z))$, which is contradiction.

Case 3. Suppose that $B=0$. Then from (8) we have

$$G = AF-(A-1). \tag{12}$$

If $A \ne 1$, then $\overline{N}(r,\frac{A-1}{A};F) = \overline{N}(r,0;G)$. Using Lemmas 1, 2, 5, 8, 9 we obtain from the Second Fundamental Theorem,

$$(n+m-\tau-\Omega)T(r,f) \le T(r,F)+S(r,f) \le \overline{N}(r,0;F)+\overline{N}\left(r,\frac{A-1}{A};F\right)+$$
$$+\overline{N}(r,\infty;F)+S(r,f) \le \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+S(r,f) \le \overline{N}(r,0;f(z))+$$
$$+N(r,0;P(f(z))+N(r,0;\prod_{j=1}^{s} f^{(j)}(z)^{\mu_j})+\overline{N}(r,0;f(qz+\eta))+N(r,0;P(f(qz+\eta))+$$
$$+N(r,0;\prod_{j=1}^{s} f^{(j)}(qz+\eta)^{\mu_j})+\overline{N}(r,\infty;f(z))+S(r,f) \le (2m+2\tau+2\Omega+3)T(r,f)+S(r,f), \tag{13}$$

i.e., $(n-m-3\tau-3\Omega-3)T(r,f) \le S(r,f)$, which contradicts that $n \ge m+3\tau+3\Omega+6$. Hence, $A=1$. From (12), we obtain $F=G$, i.e.,

$$f^n(z)P(f(z))\prod_{j=1}^{s} f^{(j)}(z)^{\mu_j} \equiv f^n(qz+\eta)P(f(qz+\eta))\prod_{j=1}^{s} f^{(j)}(qz+\eta)^{\mu_j}.$$

□

Proof of Theorem 2. Let $F = f^n(z)P(f(z))\prod_{j=1}^{s} f^{(j)}(z)^{\mu_j}$ and $G = f^n(qz+\eta)P(f(qz+\eta))\prod_{j=1}^{s} f^{(j)}(qz+\eta)^{\mu_j}$. Then F and G are nonconstant meromorphic functions that share $(1,2)$ and $(\infty,0)$. If possible, we may assume that $H \not\equiv 0$. Then we obtain from Lemma 6

$$T(r,F) \le N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_*(r,\infty;F,G)+S(r,F)+S(r,G). \tag{14}$$

As F and G share $(\infty,0)$, it is obvious that $\overline{N}_*(r,\infty;F,G) \le \overline{N}(r,\infty;F)+S(r,F)$. Therefore, using Lemmas 1, 2, 5, 8, 9 we obtain from (14)

$$(n+m-\tau-\Omega)T(r,f) \le 2\overline{N}(r,0;f(z))+N(r,0;P(f(z))+N(r,0;\prod_{j=1}^{s} f^{(j)}(z)^{\mu_j})+$$
$$+2\overline{N}(r,0;f(qz+\eta))+N(r,0;P(f(qz+\eta))+N(r,0;\prod_{j=1}^{s} f^{(j)}(qz+\eta)^{\mu_j})+$$
$$+3\overline{N}(r,\infty;f(z))+S(r,f) \le (2m+2\tau+2\Omega+7)T(r,f)+S(r,f), \tag{15}$$

i.e., $(n-m-3\tau-3\Omega-7)T(r,f)\leq S(r,f)$, which contradicts that $n\geq m+3\tau+3\Omega+7$. Therefore, we must have $H\equiv 0$ and, using similar arguments as in Theorem 1, it is obvious that Theorem 2 holds. □

Proof of Theorem 3. Let $F=f^n(z)P(f(z))\prod_{j=1}^{s}f^{(j)}(z)^{\mu_j}$ and $G=f^n(qz+\eta)P(f(qz+\eta))\prod_{j=1}^{s}f^{(j)}(qz+\eta)^{\mu_j}$. Then F and G are nonconstant meromorphic functions such that $E_{3)}(1,F)=E_{3)}(1,G)$ and share (∞,∞). If it is possible, then we may assume that $H\not\equiv 0$. Then we obtain from Lemma 7

$$\begin{aligned}T(r,F)+T(r,G)\leq{}& 2N_2(r,0;F)+2N_2(r,0;G)+2\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)\\&+2\overline{N}_*(r,\infty;F,G)+S(r,F)+S(r,G).\end{aligned}\tag{16}$$

As F and G share (∞,∞), it is obvious that $\overline{N}_*(r,\infty;F,G)=S(r,F)$. Therefore, by Lemmas 1, 2, 5, 8, 9, we obtain from (16)

$$\begin{aligned}(2n+2m-2\tau-2\Omega)T(r,f)\leq{}& 4\overline{N}(r,0;f(z))+2N(r,0;P(f(z))+2N(r,0;\prod_{j=1}^{s}f^{(j)}(z)^{\mu_j})+\\&+4\overline{N}(r,0;f(qz+\eta))+2N(r,0;P(f(qz+\eta))+2N(r,0;\prod_{j=1}^{s}f^{(j)}(qz+\eta)^{\mu_j})+\\&+4\overline{N}(r,\infty;f(z))+S(r,f)\leq(4m+4\tau+4\Omega+12)T(r,f)+S(r,f),\end{aligned}\tag{17}$$

i.e., $(2n-2m-6\tau-6\Omega-12)T(r,f)\leq S(r,f)$, which contradicts that $n\geq m+3\tau+3\Omega+6$. Therefore, we must have $H\equiv 0$ and similar arguments as in Theorem 1, we see that Theorem 3 holds. □

Proof of Theorem 4. Let us denote

$F=f^n(z)P(f(z))\prod_{j=1}^{s}f^{(j)}(z)^{\mu_j}$ and $G=f^n(qz+\eta)P(f(qz+\eta))\prod_{j=1}^{s}f^{(j)}(qz+\eta)^{\mu_j}$.

Then F and G are nonconstant meromorphic functions such that $E_{3)}(1,F)=E_{3)}(1,G)$ and share $(\infty,0)$. If it is possible, then we may assume that $H\not\equiv 0$. Then we obtain from Lemma 7

$$\begin{aligned}T(r,F)+T(r,G)\leq{}& 2N_2(r,0;F)+2N_2(r,0;G)+2\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)+\\&+2\overline{N}_*(r,\infty;F,G)+S(r,F)+S(r,G).\end{aligned}\tag{18}$$

As F and G share $(\infty,0)$, it is obvious that $\overline{N}_*(r,\infty;F,G)\leq\overline{N}(r,\infty;F)+S(r,F)$. Therefore, using Lemmas 1, 2, 5, 8 and 9 we obtain from (18)

$$\begin{aligned}(2n+2m-2\tau-2\Omega)T(r,f)\leq{}& 4\overline{N}(r,0;f(z))+2N(r,0;P(f(z))+2N(r,0;\prod_{j=1}^{s}f^{(j)}(z)^{\mu_j})+\\&+4\overline{N}(r,0;f(qz+\eta))+2N(r,0;P(f(qz+\eta))+2N(r,0;\prod_{j=1}^{s}f^{(j)}(qz+\eta)^{\mu_j})+\\&+6\overline{N}(r,\infty;f(z))+S(r,f)\leq(4m+4\tau+4\Omega+14)T(r,f)+S(r,f).\end{aligned}\tag{19}$$

i.e., $(2n-2m-6\tau-6\Omega-14)T(r,f)\leq S(r,f)$, which contradicts that $n\geq m+3\tau+3\Omega+7$. Therefore, we must have $H\equiv 0$ and similar arguments as in Theorem 1, we see that Theorem 4 holds. □

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