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A QUESTION ON THE SPECTRA OF ALGEBRAS OF SYMMETRIC FUNCTIONS ON L_∞ RELATED TO THE MOMENT PROBLEM

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We consider a question on description of the set of characters of the algebra of bounded type symmetric analytic functions on $L_\infty[0,1]$ and establish some connection with the trigonometric moment problem.

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Рассматривается вопрос описания множества характеров алгебры симметрических аналитических функций ограниченного типа на $L_\infty[0,1]$ и установлена связь с тригонометрической проблемой моментов.

1. The problem. Let $H_{bs}(L_\infty)$ be the algebra of symmetric analytic functions of bounded type on the complex $L_\infty[0,1]$. Here "symmetric" means invariant with respect to measurable transformations of $[0,1]$ preserving the Lebesgue measure. It is known (see [4]) that the functions $R_n(x) = \int_{[0,1]}(x(t))^n\,dt$, $x(t) \in L_\infty[0,1]$ $(n \in \mathbb{N})$ form an algebraic basis in the algebra of all symmetric polynomials $\mathcal{P}_s(L_\infty) \subset H_{bs}(L_\infty)$ and that $\mathcal{P}_s(L_\infty)$ is a dense subspace of $H_{bs}(L_\infty)$. Hence, if ϕ is a complex homomorphism (i.e. a character) on $H_{bs}(L_\infty)$, then ϕ is completely defined by its values on R_n $(n \in \mathbb{N})$. Since point evaluation functionals $\delta_x(f) = f(x)$, $f \in H_{bs}(L_\infty)$, $x \in L_\infty[0,1]$ are characters, we have the following natural question.

Problem 1. *Describe the set of all sequences of complex numbers $\{a_n\}_{n=1}^\infty$ such that $R_n(x) = \int_{[0,1]}(x(t))^n\,dt = a_n$ for some $x \in L_\infty[0,1]$.*

2. Partial solutions and relations to the moment problem. In [5] it is proved that, for any finite sequence $\{a_n\}_{n=1}^m$ there is a function $x(\cdot) \in L_\infty[0,1]$ such that $R_n(x) = a_n$ $n \in \{1,2,\ldots,m\}$ and $R_n(x) = 0$ for $n > m$.

Let us recall that a nondecreasing function $\sigma(\theta)$ is a solution of the trigonometric moment problem for a given sequence $\{c_k\}_{k=-\infty}^\infty$, $c_{-k} = \overline{c_k}$, $c_0 \in \mathbb{R}$ if

$$c_k = \frac{1}{2\pi}\int_{[-\pi,\pi]} \exp(ik\theta)\,d\sigma(\theta), \quad k \in \mathbb{Z}.$$

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It is known (see e.g. [1, Theorem 5.1.2]) that the trigonometric moment problem has a solution if and only if the Hermitian forms

$$\omega_n(\xi) = \sum_{\alpha,\beta=0}^{n} c_{\alpha-\beta}\xi_\alpha\overline{\xi_\beta},$$

$\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$ are nonnegative functions defined for all $n \in \mathbb{N}$. Making some simple calculations we can get the following result.

Theorem 1. *Let $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$. If $\sigma(\theta)$ is a strictly monotone solution of the trigonometric moment problem for $\{c_n\}_{n=-\infty}^{\infty}$, $c_n = a_n$ for $n > 0$, $c_0 = 1$ and $c_{-n} = \overline{a_n}$, then $x_\sigma = \exp(i\sigma^{-1}(2\pi t - \pi))$ satisfies $R_n(x_\sigma) = a_n$ $(n \in \mathbb{N})$.*

Corollary 1. *If $\{b_n\}_{n=1}^{\infty} \subset \mathbb{C}$ is such that for a given fixed number $z \in \mathbb{C}$, $\{a_n\}_{n=1}^{\infty} = \{b_n/z^n\}_{n=1}^{\infty}$ satisfies the condition of Theorem 1, then $x(t) = z\exp(i\sigma^{-1}(2\pi t - \pi))$ satisfies $R_n(x) = b_n$ $(n \in \mathbb{N})$.*

It is not difficult to construct a function $x(t) \in L_\infty[0,1]$ such that $\{b_n\}_{n=1}^{\infty} = \{R_n(x)\}_{n=1}^{\infty}$ does not satisfy the conditions of Corollary 1.

A discrete analogue of Problem 1, namely, the problem to describe all sequences $\{a_n\}_{n=1}^{\infty}$ such that $F_n(x) = a_n$, where $x = (x_1, \ldots, x_n, \ldots) \in \ell_1$ and $F_n(x) = \sum_{k=1}^{\infty} x_k^n$, was investigated in [2, 3] and is related to the problem of description of the set of all characters on the algebra $H_{bs}(\ell_1)$ of symmetric analytic functions of bounded type on ℓ_1. Note that the discrete case is quite different from the continuous one. In particular, in [2] it is proved that, if for some $m > 0$, $F_n(x) = 0$ for all $n > m$, then $x = 0$, while there is a character ψ on $H_{bs}(\ell_1)$ such that $\psi(F_1) = 1$ and $\psi(F_n) = 0$ for $n > 1$. From results of [3] it follows that for every $x \in \ell_1$, $\sum_{n=0}^{\infty} G_n(z)z^n$ is a function of exponential type with zeros at $z_n = -1/x_n$ for all $x_n \neq 0$, where $G_0(z) = 1$, $G_1(z) = F_1(z)$, $nG_n(z) = F_1(z)G_{n-1}(z) - F_2(z)G_{n-2}(z) + \ldots + (-1)^{n-1}F_n(z)$, $n > 1$. However, the problem about description of all characters on $H_{bs}(\ell_1)$ is still open.

REFERENCES

1. N.I. Akhiezer, The classical moment problem and some related questions in analysis, Hafner Publishing Co., New York, 1965.
2. R. Alencar, R. Aron, P. Galindo, A. Zagorodnyuk, *Algebras of symmetric holomorphic functions on ℓ_p*, Bull. London Math. Soc., **35** (2003), 55–64.
3. I. Chernega, P. Galindo, A. Zagorodnyuk, *The convolution operation on the spectra of algebras of symmetric analytic functions,* J. Math. Anal. Appl., **395** (2012), 569–577.
4. M. Gonzales, R. Gonzalo, J. A. Jaramillo, *Symmetric polynomials on rearrangement invariant function spaces*, J. London Math. Soc. (2), **59** (1999), 681–697.
5. T. Vasylyshyn, *Construction of complex-valued functions by pre-defined values of integrals of their powers*, Mathematical Bulletin of the Shevchenko Scientific Soc., **8** (2011), 17–27.

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