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## ON TRUNCATION ERROR BOUNDS OF BRANCHED CONTINUED FRACTION EXPANSIONS OF SOME RATIOS OF LAURICELLA–SARAN’S HYPERGEOMETRIC FUNCTIONS $F_K$

R. Dmytryshyn, V. Goran. *On truncation error bounds of branched continued fraction expansions of some ratios of Lauricella–Saran’s hypergeometric functions  $F_K$* , *Mat. Stud.* **65** (2026), 48–57.

The paper considers the problem of approximating Lauricella–Saran’s hypergeometric functions  $F_K$  in special cases by branched continued fractions as a special family of functions. Under the certain conditions on the elements of branched continued fraction expansions of some ratios of these functions, it is proven that every expansion converges to the function that is analytic in the domain of analytic continuation

$$\mathfrak{D}_\eta = \{\mathbf{z} \in \mathbb{R}^3 : z_1 \leq \eta, z_2 \leq \eta, z_3 \leq 0\}, \quad 0 < \eta < 1,$$

at least as fast as a geometric series with a ratio less than unity. For this purpose, the method based on the formula for the difference of two approximants of a branched continued fraction and the PF method (based on the so-called property of fork for a branched continued fraction with positive elements) was used.

**1. Introduction.** Branched continued fractions as a class of special functions are one of the most effective tools for representing the analytic functions that naturally arise in a variety of applied problems (see, for example, [16, 17, 18] and also [35, 42]). The expansions of some ratios of the Lauricella–Saran’s hypergeometric functions  $F_K$  into branched continued fractions were constructed and investigated in [2, 23]. This was done on the base of three-term recurrence relations ([2, Lemma 1]), using the idea of choosing certain identical values of the parameters of hypergeometric functions, proposed in [4]. The need to compute the values of the function  $F_K$  in special cases arises, in particular, in the problem of studying of the propagator seagull diagram [36]. Other applications of  $F_K$  can be found in [2, 15, 34] and also [39]. In the same way, the long-standing problem of studying the convergence of the branched continued fraction expansion of one ratio of Appel’s hypergeometric functions  $F_2$  (see [14]) was partially solved in a special case in [1, 24, 26]. And recently, in [25, 40], two expansions of special ratios of the Lauricella–Saran’s hypergeometric functions  $F_M$  were constructed and investigated.

This paper establishes the estimates of convergence rate of the above-mentioned expansions for the ratios of Lauricella–Saran’s hypergeometric functions  $F_K$  in the special cases.

2020 *Mathematics Subject Classification*: 33C65; 32A17; 41A20; 32A10; 40A99; 30B40.

*Keywords*: hypergeometric function; branched continued fraction; convergence; analytic continuation.

doi:10.30970/ms.65.1.48-57

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Such estimates (also called truncation error bounds [25]) are important from a computational point view. Truncation error bounds were established for branched continued fraction expansions of some ratios of generalized hypergeometric functions  ${}_3F_2$  in [5], Appell's hypergeometric functions  $F_1$  in [27],  $F_2$  in [26],  $F_4$  in [30], Horn's hypergeometric functions  $H_3$  in [3],  $H_4$  in [21], Horn's confluent hypergeometric functions  $H_6$  in [6], and Lauricella–Saran's hypergeometric functions  $F_M$  in [25]. Each of them is based on the formula for the difference of two approximants of a branched continued fraction (see [13, p. 28]), adapted to the constructed expansions (for more on branched continued fraction structures, see [9]). More about the convergence problem can be found in [10, 12] and also in [20, 32].

**2. Expansions of some ratios of hypergeometric functions  $F_K$ .** The function  $F_K$  is defined by triple power series (see [37, 41])

$$F_K(\alpha_1, \alpha_2, \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; \mathbf{z}) = \sum_{p,q,r=0}^{+\infty} \frac{(\alpha_1)_p (\alpha_2)_{q+r} (\beta_1)_{p+r} (\beta_2)_q}{(\gamma_1)_p (\gamma_2)_q (\gamma_3)_r} \frac{z_1^p z_2^q z_3^r}{p! q! r!}, \quad (1)$$

where  $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3 \in \mathbb{C}$  herewith  $\gamma_1, \gamma_2, \gamma_3 \notin \mathbb{Z}_{\leq}$ ,  $\mathbb{Z}_{\leq}$  denotes the nonpositive integers,  $\mathbf{z} = (z_1, z_2, z_3) \in \mathfrak{D}_{F_K}$ ,

$$\mathfrak{D}_{F_K} = \{\mathbf{z} \in \mathbb{C}^3 : |z_1| < 1, |z_2| < 1, |z_3| < (1 - |z_1|)(1 - |z_2|)\},$$

$(\xi)_k = \xi(\xi + 1) \cdots (\xi + k - 1)$  is the Pochhammer symbol.

In [2, Theorem 3] the following is proved.

**Theorem A.** *The special ratio*

$$\frac{F_K(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1, \beta_2, \gamma_3; \mathbf{z})}{F_K(\alpha_1, \alpha_2, \beta_1 + 1, \beta_2; \alpha_1, \beta_2, \gamma_3 + 1; \mathbf{z})} \quad (2)$$

has a formal branched continued fraction expansion

$$1 - z_1 - \frac{\nu_1 z_3}{1 - z_2 - \frac{\nu_2 z_3}{1 - z_1 - \frac{\nu_3 z_3}{1 - z_2 - \frac{\nu_4 z_3}{1 - \dots}}}}, \quad (3)$$

where  $\alpha_1, \beta_2, \gamma_3 \notin \mathbb{Z}_{\leq}$ , and

$$\nu_{2n-1} = \frac{(\alpha_2 + n - 1)(\gamma_3 + n - 1 - \beta_1)}{(\gamma_3 + 2n - 2)(\gamma_3 + 2n - 1)}, \quad \nu_{2n} = \frac{(\beta_1 + n)(\gamma_3 + n - \alpha_2)}{(\gamma_3 + 2n - 1)(\gamma_3 + 2n)}, \quad n \geq 1. \quad (4)$$

The following result can be proved in much the same way.

**Theorem B.** *The special ratio*

$$\frac{F_K(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1, \beta_2, \gamma_3; \mathbf{z})}{F_K(\alpha_1, \alpha_2 + 1, \beta_1, \beta_2; \alpha_1, \beta_2, \gamma_3 + 1; \mathbf{z})} \quad (5)$$

has a formal branched continued fraction expansion

$$1 - z_2 - \frac{\mu_1 z_3}{1 - z_1 - \frac{\mu_2 z_3}{1 - z_2 - \frac{\mu_3 z_3}{1 - z_1 - \frac{\mu_4 z_3}{1 - \dots}}}}, \quad (6)$$

where  $\alpha_1, \beta_2, \gamma_3 \notin \mathbb{Z}_{\leq}$ , and

$$\mu_{2n-1} = \frac{(\beta_1 + n - 1)(\gamma_3 + n - 1 - \alpha_2)}{(\gamma_3 + 2n - 2)(\gamma_3 + 2n - 1)}, \quad \mu_{2n} = \frac{(\alpha_2 + n)(\gamma_3 + n - \beta_1)}{(\gamma_3 + 2n - 1)(\gamma_3 + 2n)}, \quad n \geq 1. \quad (7)$$

The ratios (2) and (5) form the so-called complete group of ratios of the hypergeometric function  $F_K(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1, \beta_2, \gamma_3; \mathbf{z})$  (see [38, Definition 1]).

Let  $\alpha_1, \beta_2$  be complex numbers herewith  $\alpha_1, \beta_2 \notin \mathbb{Z}_{\leq}$  and  $\alpha_2, \beta_1, \gamma_3$  be real numbers. In [2, Theorem 6] it is shown that under the conditions

$$0 < \nu_n \leq \tau, \quad n \geq 1, \quad \tau > 0. \quad (8)$$

the branched continued fraction (3) converges uniformly on every compact subset of the domain

$$\mathfrak{D}_{\tau, \kappa} = \left\{ \mathbf{z} \in \mathbb{C}^3 : z_n \notin [1 - \kappa, +\infty), \quad n = 1, 2, \quad z_3 \notin \left[ \frac{\kappa}{4\tau}, +\infty \right) \right\}, \quad 0 < \kappa < 1, \quad (9)$$

to a function  $f(\mathbf{z})$  that is holomorphic in  $\mathfrak{D}_{\tau, \kappa}$ , and, in addition, the function  $f(\mathbf{z})$  is an analytic continuation of the function (2) in the domain (8). A similar result is valid for the function (5) and its expansion (6) provided that

$$0 < \mu_n \leq \tau, \quad n \geq 1, \quad \tau > 0. \quad (10)$$

In [23, Theorem 2], under conditions that  $\alpha_1, \alpha_2, \beta_1, \beta_2$ , and  $\gamma_3$  are complex constants such that  $\alpha_1, \beta_2 \notin \mathbb{Z}_{\leq}$  and

$$|\nu_n| + \operatorname{Re}(\nu_n) \leq pq(1 - q), \quad n \geq 1, \quad p > 0, \quad 0 < q < 1,$$

the uniformly convergence of (3) on every compact subset of the domain

$$\mathfrak{D}_{p, q}^{\nu, \kappa} = \mathfrak{D}_{p, q} \bigcup \mathfrak{D}^{\nu, \kappa}, \quad (11)$$

to a function  $f(\mathbf{z})$  holomorphic in  $\mathfrak{D}_{p, q}^{\nu, \kappa}$  is proved, and, in addition, it is shown that  $f(\mathbf{z})$  is an analytic continuation of (3) in  $\mathfrak{D}_{p, q}^{\tau, \kappa}$ , where

$$\mathfrak{D}_{p, q} = \left\{ \mathbf{z} \in \mathbb{C}^3 : z_n \neq \left[ \frac{q}{2}, +\infty \right), \quad n = 1, 2, \quad |z_3| < \frac{1 + \cos(\arg(z_3))}{2p} \right\}$$

and

$$\mathfrak{D}^{\nu, \kappa} = \left\{ \mathbf{z} \in \mathbb{C}^3 : |z_n| < \frac{1 - \kappa}{2}, \quad n = 1, 2, \quad |z_3| < \frac{\kappa(1 - \kappa)}{2\nu} \right\}, \quad \nu = \sup_{n \in \mathbb{N}} \{|\nu_n|\}, \quad 0 < \kappa < 1.$$

Also, a similar result is valid for (5) and (6) provided that  $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_3 \in \mathbb{C}$ ,  $\alpha_1, \beta_2 \notin \mathbb{Z}_{\leq}$ , and

$$|\mu_n| + \operatorname{Re}(\mu_n) \leq pq(1 - q), \quad n \geq 1, \quad p > 0, \quad 0 < q < 1,$$

where  $\mu_n$ ,  $n \geq 1$ , are defined by (7).

**3. Main results.** Let  $\{f_n(\mathbf{z})\}$  be a sequence of approximants of branched continued fraction (3). We set

$$U_n^{(n)}(\mathbf{z}) = 1, \quad n \geq 1, \quad (12)$$

and, for  $n \geq 1$ ,  $1 \leq k \leq n$ ,

$$U_k^{(n)}(\mathbf{z}) = 1 - z_{1+\delta_1^{(-1)k+1}} - \frac{\nu_{k+1}z_3}{1 - z_{1+\delta_1^{(-1)k+2}} - \frac{\nu_{k+2}z_3}{1 - \dots - z_{1+\delta_1^{(-1)n-1}} - \frac{\nu_{n-1}z_3}{1 - z_{1+\delta_1^{(-1)n}} - \nu_n z_3}},$$

where  $\delta_i^j$  is a Kronecker symbol,  $\nu_n$ ,  $n \geq 1$ , are defined by (4). Then, for  $n \geq 2$ ,  $1 \leq k \leq n-1$ ,

$$U_k^{(n)}(\mathbf{z}) = 1 - z_{1+\delta_1^{(-1)k+1}} - \frac{\nu_{k+1}z_3}{U_{k+1}^{(n)}(\mathbf{z})} \quad (13)$$

and  $f_n(\mathbf{z}) = 1 - z_1 - \frac{\nu_1 z_3}{U_1^{(n)}(\mathbf{z})}$ ,  $n \geq 1$ .

Assume that  $U_k^{(n)}(\mathbf{z}) \neq 0$  for all indices in some domain  $\mathfrak{D}$ ,  $\mathfrak{D} \subset \mathbb{C}^3$ . Then for  $n \geq 1$ ,  $k \geq 1$  (see [13, p. 28])

$$f_{n+k}(\mathbf{z}) - f_n(\mathbf{z}) = -z_3^n \left( z_{1+\delta_1^{(-1)n+1}} + \frac{\nu_{n+1}z_3}{U_{n+1}^{(n+k)}(\mathbf{z})} \right) \prod_{r=1}^n \frac{\nu_r}{U_r^{(n+k)}(\mathbf{z})U_r^{(n)}(\mathbf{z})} \quad (14)$$

or for  $n \geq 2$ ,  $k \geq 2$

$$f_{n+k}(\mathbf{z}) - f_n(\mathbf{z}) = \frac{-\nu_1 z_3^n}{U_1^{(q)}(\mathbf{z})U_n^{(n+k)}(\mathbf{z})} \left( z_{1+\delta_1^{(-1)n+1}} + \frac{\nu_{n+1}z_3}{U_{n+1}^{(n+k)}(\mathbf{z})} \right) \prod_{r=1}^{[n/2]} \frac{\nu_{2r}}{U_{2r-1}^{(p)}(\mathbf{z})U_{2r}^{(p)}(\mathbf{z})} \prod_{r=1}^{[(n-1)/2]} \frac{\nu_{2r+1}}{U_{2r}^{(q)}(\mathbf{z})U_{2r+1}^{(q)}(\mathbf{z})},$$

where  $[\cdot]$  denote integer part,  $q = n+k$ ,  $p = n$ , if  $n = 2s$ , and  $q = n$ ,  $p = n+k$ , if  $n = 2s-1$ ,  $s \geq 1$ .

The following result holds.

**Theorem 1.** Let  $\alpha_1, \beta_2$  be complex number herewith  $\alpha_1, \beta_2 \notin \mathbb{Z}_{\leq}$  and  $\alpha_2, \beta_1$ , and  $\gamma_3$  real numbers such that satisfy the conditions (8). Then:

(a) the branched continued fraction (3) converges to a finite value  $f(\mathbf{z})$  for each  $\mathbf{z} \in \mathfrak{D}_\eta$ , where

$$\mathfrak{D}_\eta = \{\mathbf{z} \in \mathbb{C}^3: z_1 \leq \eta, z_2 \leq \eta, z_3 \leq 0\}, \quad 0 < \eta < 1; \quad (15)$$

(b) the convergence is uniform on every compact subset of the domain  $\text{Int}(\mathfrak{D}_\eta)$ , and the function  $f(\mathbf{z})$  is analytic on  $\text{Int}(\mathfrak{D}_\eta)$ ;

(c) if  $f_n(\mathbf{z})$  denotes the  $n$ th approximant of (3), then for  $n \geq 2$  and for each  $\mathbf{z} \in \mathfrak{D}_\eta$

$$\begin{aligned} & |f(\mathbf{z}) - f_n(\mathbf{z})| \leq \\ & \leq \frac{(|z_{1+\delta_1^{(-1)n+1}}|(1 - z_{1+\delta_1^{(-1)n+2}}) + \tau|z_3|)(1 - z_{1+\delta_1^{(-1)n+1}} + \tau|z_3|)^{-1}(\tau|z_3|)^n}{(1 - z_1)(1 - z_2)(1 - z_1 + \tau|z_3|)^{[(n-1)/2]}(1 - z_2 + \tau|z_3|)^{[n/2]}}; \end{aligned} \quad (16)$$

(d) the function  $f(\mathbf{z})$  is an analytic continuation of the function (2) in the domain  $\text{Int}(\mathfrak{D}_\eta)$ .

*Proof.* (a) Let  $\mathbf{z}$  be an arbitrary fixed point in (15). Using inequalities (8), from (13) for arbitrariness  $n$  and  $k$  such that  $1 \leq k \leq n-1$  and  $n \geq 2$ , we have

$$U_k^{(n)}(\mathbf{z}) = 1 - z_{1+\delta_1^{(-1)k+1}} - \frac{\nu_{k+1}z_3}{U_{k+1}^{(n)}(\mathbf{z})} \geq 1 - z_{1+\delta_1^{(-1)k+1}} \geq 1 - \eta > 0. \quad (17)$$

Now, using (8), (15), and (17), for any  $n \geq 2$  and  $k \geq 2$ , we have that

$$|-\nu_1 z_3|/|U_1^{(n)}(\mathbf{z})| \leq \tau|z_3|$$

and

$$\frac{1}{|U_n^{(n+k)}(\mathbf{z})|} \cdot \left| z_{1+\delta_1^{(-1)n+1}} + \frac{\nu_{n+1} z_3}{U_{n+1}^{(n+k)}(\mathbf{z})} \right| \leq \frac{1}{1 - z_{1+\delta_1^{(-1)n+1}}} \left( |z_{1+\delta_1^{(-1)n+1}}| + \frac{\tau|z_3|}{1 - z_{1+\delta_1^{(-1)n+2}}} \right).$$

Next, additionally using (13) and for  $1 \leq k \leq n-1$  and  $n \geq 2$ , we obtain

$$\begin{aligned} \left| \frac{\nu_{k+1} z_3}{U_k^{(n+1)}(\mathbf{z}) U_{k+1}^{(n+1)}(\mathbf{z})} \right| &= \frac{\left| \frac{\nu_{k+1} z_3}{U_{k+1}^{(n+1)}(\mathbf{z})} \right|}{\left| 1 - z_{1+\delta_1^{(-1)k+1}} - \frac{\nu_{k+1} z_3}{U_{k+1}^{(n+1)}(\mathbf{z})} \right|} \leq \frac{\frac{\nu_{k+1} |z_3|}{U_{k+1}^{(n+1)}(\mathbf{z})}}{1 - z_{1+\delta_1^{(-1)k+1}} + \frac{\nu_{k+1} |z_3|}{U_{k+1}^{(n+1)}(\mathbf{z})}} \leq \\ &\leq \frac{\tau|z_3|}{1 - z_{1+\delta_1^{(-1)k+1}} + \tau|z_3|} \end{aligned}$$

and, furthermore,

$$\left| \frac{\nu_{n+1} z_3}{U_n^{(n+1)}(\mathbf{z}) U_{n+1}^{(n+1)}(\mathbf{z})} \right| = \frac{\nu_{n+1} z_3}{1 - z_{1+\delta_1^{(-1)n+1}} - \nu_{n+1} z_3} \leq \frac{\tau|z_3|}{1 - z_{1+\delta_1^{(-1)n+1}} + \tau|z_3|}.$$

Due to the above, for  $n \geq 2$  and  $k \geq 2$ , we have the following

$$\begin{aligned} &|f_{n+k}(\mathbf{z}) - f_n(\mathbf{z})| \leq \\ &\leq \frac{(|z_{1+\delta_1^{(-1)n+1}}|(1 - z_{1+\delta_1^{(-1)n+2}}) + \tau|z_3|)(1 - z_{1+\delta_1^{(-1)n+1}} + \tau|z_3|)^{-1}(\tau|z_3|)^n}{(1 - z_{1+\delta_1^{(-1)n+1}})(1 - z_{1+\delta_1^{(-1)n+2}})(1 - z_1 + \tau|z_3|)^{[(n-1)/2]}(1 - z_2 + \tau|z_3|)^{[n/2]}} = \\ &= \frac{(|z_{1+\delta_1^{(-1)n+1}}|(1 - z_{1+\delta_1^{(-1)n+2}}) + \tau|z_3|)(\tau|z_3|)^n}{(1 - z_1)(1 - z_2)(1 - z_{1+\delta_1^{(-1)n+1}} + \tau|z_3|)(1 - z_1 + \tau|z_3|)^{[(n-1)/2]}(1 - z_2 + \tau|z_3|)^{[n/2]}}. \quad (18) \end{aligned}$$

Obviously, for an arbitrary fixed  $\mathbf{z} \in \mathfrak{D}_\eta$ ,

$$\frac{(|z_{1+\delta_1^{(-1)n+1}}|(1 - z_{1+\delta_1^{(-1)n+2}}) + \tau|z_3|)(\tau|z_3|)^n}{(1 - z_1)(1 - z_2)(1 - z_{1+\delta_1^{(-1)n+1}} + \tau|z_3|)(1 - z_1 + \tau|z_3|)^{[(n-1)/2]}(1 - z_2 + \tau|z_3|)^{[n/2]}} \rightarrow 0$$

as  $n \rightarrow +\infty$ . Thus, due to the arbitrariness of  $k$ , we get (a).

(b) Let  $\mathfrak{K}$  be an arbitrary compact subset of  $\text{Int}(\mathfrak{D}_\eta)$ . Then there exists a constant  $C > 0$  such that for  $n \geq 2$ ,  $k \geq 2$ , and for all  $\mathbf{z} \in \mathfrak{K}$  from (18) we obtain

$$|f_{n+k}(\mathbf{z}) - f_n(\mathbf{z})| < \frac{(C(1 - \eta) + \tau C)(\tau C)^n}{(1 - \eta)^3(1 - \eta + \tau C)^{n-1}}.$$

Now, if  $m, n$  are arbitrary integer numbers such that  $m \geq 2$ ,  $n + 2 \geq k \geq 2$ , then, for all  $\mathbf{z} \in \mathfrak{K}$ ,

$$|f_{n+m}(\mathbf{z}) - f_n(\mathbf{z})| \leq |f_{n+m}(\mathbf{z}) - f_k(\mathbf{z})| + |f_n(\mathbf{z}) - f_k(\mathbf{z})|.$$

Due to

$$\frac{(C(1 - \eta) + \tau C)(\tau C)^n}{(1 - \eta)^3(1 - \eta + \tau C)^{n-1}} \rightarrow 0 \quad (n \rightarrow +\infty),$$

(b) follows.

(c) follows directly from (18) as  $k \rightarrow +\infty$ .

(d) It is obvious that

$$\frac{F_K(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1, \beta_2, \gamma_3; \mathbf{0})}{F_K(\alpha_1, \alpha_2, \beta_1 + 1, \beta_2; \alpha_1, \beta_2, \gamma_3 + 1; \mathbf{0})} = 1.$$

Then, there exists  $0 < \varepsilon < 1/4$  such that function (2) is analytic in domain

$$\mathfrak{D}_\varepsilon = \{\mathbf{z} \in \mathbb{R}^3: -\varepsilon < z_1 < 0, -\varepsilon < z_2 < 0, -\varepsilon < z_3 < 0\},$$

and  $\mathfrak{D}_\varepsilon \subset (\mathfrak{D}_{F_K} \cap \text{Int}(\mathfrak{D}_\eta))$ , in particular,  $\mathfrak{D}_{1/8} \subset (\mathfrak{D}_{F_K} \cap \text{Int}(\mathfrak{D}_\eta))$ .

Let  $\mathbf{z}$  be an arbitrary fixed point in  $\mathfrak{D}_\varepsilon$ . Then it is obvious that the elements of expansion (3) are positive, which gives us the following (see, [13, Proposition 1.4])

$$f_{2n}(\mathbf{z}) < f_{2n+2}(\mathbf{z}) < f_{2n+1}(\mathbf{z}) < f_{2n-1}(\mathbf{z}), \quad n \geq 1.$$

This, together with (a), for each  $\mathbf{z} \in \mathfrak{D}_\varepsilon$  ensures the convergence of the sequences of even and odd approximants of branched continued fraction (3) to a finite value  $f(\mathbf{z})$ .

Now, we consider

$$\frac{F_K(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1, \beta_2, \gamma_3; \mathbf{z})}{F_K(\alpha_1, \alpha_2, \beta_1 + 1, \beta_2; \alpha_1, \beta_2, \gamma_3 + 1; \mathbf{z})} - f_n(\mathbf{z}), \quad n \geq 1,$$

where

$$\begin{aligned} & \frac{F_K(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1, \beta_2, \gamma_3; \mathbf{z})}{F_K(\alpha_1, \alpha_2, \beta_1 + 1, \beta_2; \alpha_1, \beta_2, \gamma_3 + 1; \mathbf{z})} = \\ & = 1 - z_2 - \frac{\nu_1 z_3}{1 - z_1 - \frac{\nu_2 z_3}{1 - \dots - z_{1+\delta_1^{(-1)n}} - \frac{\nu_n z_3}{1 - z_{1+\delta_1^{(-1)n+1}} - \frac{\nu_{n+1} z_3}{V_{n+1}^{(n+1)}(\mathbf{z})}}}}}, \end{aligned}$$

and

$$V_{n+1}^{(n+1)}(\mathbf{z}) = \frac{F_K\left(\alpha_1, \alpha_2 + \sum_{r=0}^n \delta_1^{(-1)r}, \beta_1 + \sum_{r=0}^{n-1} \delta_1^{(-1)r}, \beta_2; \alpha_1, \beta_2, \gamma_3 + n + 1; \mathbf{z}\right)}{F_K\left(\alpha_1, \alpha_2 + \sum_{r=0}^{n+1} \delta_1^{(-1)r}, \beta_1 + \sum_{r=0}^n \delta_1^{(-1)r}, \beta_2; \alpha_1, \beta_2, \gamma_3 + n + 2; \mathbf{z}\right)}.$$

Let

$$\begin{aligned} & V_k^{(n+1)}(\mathbf{z}) = \\ & = 1 - z_{1+\delta_1^{(-1)k+1}} - \frac{\nu_{k+1} z_3}{1 - z_{1+\delta_1^{(-1)k+2}} - \frac{\nu_{k+2} z_3}{1 - \dots - z_{1+\delta_1^{(-1)n}} - \frac{\nu_n z_3}{1 - z_{1+\delta_1^{(-1)n+1}} - \frac{\nu_{n+1} z_3}{V_{n+1}^{(n+1)}(\mathbf{z})}}}}}. \end{aligned}$$

Then, similarly to (13), for  $n \geq 2$ ,  $1 \leq k \leq n-1$  we have

$$V_k^{(n)}(\mathbf{z}) = 1 - z_{1+\delta_1^{(-1)k+1}} - \frac{\nu_{k+1} z_3}{V_{k+1}^{(n)}(\mathbf{z})}.$$

Obviously,  $U_k^{(n)}(\mathbf{z}) \neq 0$ ,  $V_k^{(n)}(\mathbf{z}) \neq 0$  for all indices and for all  $\mathbf{z} \in \mathfrak{D}_\varepsilon$ . Then, taking (14) into account for  $n \geq 1$ , we obtain

$$\begin{aligned} & \frac{F_K(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1, \beta_2, \gamma_3; \mathbf{z})}{F_K(\alpha_1, \alpha_2, \beta_1 + 1, \beta_2; \alpha_1, \beta_2, \gamma_3 + 1; \mathbf{z})} - f_n(\mathbf{z}) = \\ & = -z_3^n \left( z_{1+\delta_1^{(-1)n+1}} + \frac{\nu_{n+1} z_3}{V_{n+1}^{(n+1)}(\mathbf{z})} \right) \prod_{r=1}^n \frac{\nu_r}{V_r^{(n+1)}(\mathbf{z}) U_r^{(n)}(\mathbf{z})}. \end{aligned}$$

Therefore, for  $n \geq 1$  and for all  $\mathbf{z} \in \mathfrak{D}_\varepsilon$ ,

$$f_{2n}(\mathbf{z}) < \frac{F_K(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1, \beta_2, \gamma_3; \mathbf{z})}{F_K(\alpha_1, \alpha_2, \beta_1 + 1, \beta_2; \alpha_1, \beta_2, \gamma_3 + 1; \mathbf{z})} < f_{2n-1}(\mathbf{z}).$$

Now, due to the above, including (a), we get  $\lim_{n \rightarrow +\infty} f_{2n}(\mathbf{z}) = \lim_{n \rightarrow +\infty} f_{2n-1}(\mathbf{z}) = f(\mathbf{z})$ . Then also for all  $\mathbf{z} \in \mathfrak{D}_\varepsilon$ ,

$$f(\mathbf{z}) = \frac{F_K(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1, \beta_2, \gamma_3; \mathbf{z})}{F_K(\alpha_1, \alpha_2, \beta_1 + 1, \beta_2; \alpha_1, \beta_2, \gamma_3 + 1; \mathbf{z})}.$$

Finally, applying [2, Theorem 2], we obtain (d).  $\square$

By setting  $\beta_1 = 0$  and replacing  $\gamma_3$  with  $\gamma_3 - 1$  in Theorem 1, we obtain a similar result for the branched continued fraction

$$\frac{1}{1 - z_1 - \frac{u_1 z_3}{1 - z_2 - \frac{u_2 z_3}{1 - z_1 - \frac{u_3 z_3}{1 - z_2 - \frac{u_4 z_3}{1 - \dots}}}}}, \quad (19)$$

where

$$u_{2n-1} = \frac{(\alpha_2 + n - 1)(\gamma_3 + n - 2)}{(\gamma_3 + 2n - 3)(\gamma_3 + 2n - 2)}, \quad u_{2n} = \frac{n(\gamma_3 + n - 1 - \alpha_2)}{(\gamma_3 + 2n - 2)(\gamma_3 + 2n - 1)}, \quad n \geq 1.$$

**Corollary 1.** *If  $\alpha_1$  and  $\beta_2$  be complex numbers herewith  $\alpha_1, \beta_2 \notin \mathbb{Z}_\leq$ ,  $\alpha_2$  and  $\gamma_3$  be real numbers such that  $0 < u_n \leq \varkappa$ ,  $n \geq 1$ ,  $\varkappa > 0$ , then:*

- (a) *the branched continued fraction (19) converges to a finite value  $g(\mathbf{z})$  for each  $\mathbf{z} \in \mathfrak{D}_\eta$ , where  $\mathfrak{D}_\eta$  is defined by (15);*
- (b) *the convergence is uniform on every compact subset of (15) to the function  $g(\mathbf{z})$  analytic in  $\text{Int}(\mathfrak{D}_\eta)$ ;*
- (c) *if  $g_n(\mathbf{z})$  denotes the  $n$ th approximant of (19), then for  $n \geq 3$  and for each  $\mathbf{z} \in \mathfrak{D}_\eta$*

$$\begin{aligned} & |g(\mathbf{z}) - g_n(\mathbf{z})| \leq \\ & \leq \frac{(|z_{1+\delta_1^{(-1)^{n+1}}}|(1 - z_{1+\delta_1^{(-1)^{n+2}}}) + \varkappa|z_3|)(1 - z_{1+\delta_1^{(-1)^{n+1}}} + \varkappa|z_3|)^{-1}(\varkappa|z_3|)^{n-1}}{(1 - z_1)^3(1 - z_2)(1 - z_1 + \varkappa|z_3|)^{[(n-2)/2]}(1 - z_2 + \varkappa|z_3|)^{[(n-1)/2]}}; \end{aligned} \quad (20)$$

- d) *the function  $g(\mathbf{z})$  is an analytic continuation of  $F_K(a_1, a_2, 1, b_2; a_1, b_2, c_3; \mathbf{z})$  in  $\text{Int}(\mathfrak{D}_\eta)$ .*

The following theorem can be proved in much the same way as Theorem 1.

**Theorem 2.** *Let  $\alpha_1, \beta_2 \in \mathbb{C}$  herewith  $\alpha_1, \beta_2 \notin \mathbb{Z}_\leq$ ,  $\alpha_2, \beta_1, \gamma_3 \in \mathbb{R}$  and the inequalities (10) hold. Then:*

- (a) *the branched continued fraction (6) converges to a finite value  $f(\mathbf{z})$  for each  $\mathbf{z} \in \mathfrak{D}_\eta$ , where  $\mathfrak{D}_\eta$  is defined by (15);*
- (b) *the convergence is uniform on every compact subset of the domain  $\text{Int}(\mathfrak{D}_\eta)$ , and the function  $f(\mathbf{z})$  is analytic on  $\text{Int}(\mathfrak{D}_\eta)$ ;*
- (c) *if  $f_n(\mathbf{z})$  denotes the  $n$ th approximant of (6), then for  $n \geq 2$  and for each  $\mathbf{z} \in \mathfrak{D}_\eta$  the truncation error bound (16) holds;*
- (d) *the function  $f(\mathbf{z})$  is an analytic continuation of the function (5) in the domain  $\text{Int}(\mathfrak{D}_\eta)$ .*

**Corollary 2.** Let  $\alpha_1, \beta_2 \in \mathbb{C}$  herewith  $\alpha_1, \beta_2 \notin \mathbb{Z}_{\leq}$ ,  $\beta_1, \gamma_3 \in \mathbb{R}$  and  $0 < v_n \leq \varkappa$ ,  $n \geq 1$ ,  $\varkappa > 0$ , where

$$v_{2n-1} = \frac{(\beta_1 + n - 1)(\gamma_3 + n - 2)}{(\gamma_3 + 2n - 3)(\gamma_3 + 2n - 2)}, \quad v_{2n} = \frac{n(\gamma_3 + n - 1 - \beta_1)}{(\gamma_3 + 2n - 2)(\gamma_3 + 2n - 1)}, \quad n \geq 1.$$

Then:

(a) the branched continued fraction

$$\frac{1}{1 - z_2 - \frac{v_1 z_3}{1 - z_1 - \frac{v_2 z_3}{1 - z_2 - \frac{v_3 z_3}{1 - z_1 - \frac{v_4 z_3}{1 - \dots}}}}}, \quad (21)$$

converges to a finite value  $g(\mathbf{z})$  for each  $\mathbf{z} \in \mathfrak{D}_\eta$ , where  $\mathfrak{D}_\eta$  is defined by (15);

(b) the convergence is uniform on every compact subset of (15) to the function  $g(\mathbf{z})$  analytic in  $\text{Int}(\mathfrak{D}_\eta)$ ;

(c) if  $g_n(\mathbf{z})$  denotes the  $n$ th approximant of (21), then for  $n \geq 3$  and for each  $\mathbf{z} \in \mathfrak{D}_\eta$  the truncation error bound (20) holds;

(d) the function  $g(\mathbf{z})$  is an analytic continuation of  $F_K(a_1, 1, b_1, b_2; a_1, b_2, c_3; \mathbf{z})$  in  $\text{Int}(\mathfrak{D}_\eta)$ .

**4. Conclusions.** For each  $\mathbf{z} \in \text{Int}(\mathfrak{D}_\eta)$ , where  $\mathfrak{D}_\eta$  is defined by (15), we have established that under conditions (8) the branched continued fraction (3) converges to the function  $f(\mathbf{z})$  which is analytic in  $\text{Int}(\mathfrak{D}_\eta)$ , at least as fast as geometric series with ratio

$$\rho(\mathbf{z}) = \tau|z_3| / (\min\{1 - z_1, 1 - z_2\} + \tau|z_3|). \text{ Thus, } \overline{\lim}_{n \rightarrow +\infty} |f(\mathbf{z}) - f_n(\mathbf{z})|^{1/n} \leq \rho(\mathbf{z}),$$

where  $f_n(\mathbf{z})$  is  $n$ th approximant of (3). Using the PF method (based on the so-called property of fork for a branched continued fraction with positive elements) [2], it is also established that under conditions (8), the function  $f(\mathbf{z})$ , to which the expansion (3) converges, is an analytic extension of the function (2) in the domain (15). A similar result is valid for the function (5) and its expansion (6) provided that the inequalities (10) hold.

Further research is to establish truncation error bounds for expansions (3) and (5) in the domains of (9) and (11). For this, we can try to use the ideas of obtaining estimates in [7, 8, 11]. Another direction is to establish sets of numerical stability of these expansions. Here, we can use methods for both continued fractions [19, 31] and branched continued fractions [28, 29, 33].

**Acknowledgments.** This research was supported by the National Research Foundation of Ukraine, 2023.03/0198 “Analysis of the spectra of countably generated algebras of symmetric polynomials and possible applications in quantum mechanics and computer science”.

## REFERENCES

1. T. Antonova, C. Cesarano, R. Dmytryshyn, S. Sharyn, *An approximation to Appell's hypergeometric function  $F_2$  by branched continued fraction*, Dolomites Res. Notes Approx., **17** (2024), 22–31. doi:10.14658/PUPJ-DRNA-2024-1-3
2. T. Antonova, R. Dmytryshyn, V. Goran, *On the analytic continuation of Lauricella-Saran hypergeometric function  $F_K(a_1, a_2, b_1, b_2; a_1, b_2, c_3; \mathbf{z})$* , Mathematics, **11** (2023), 4487. doi:10.3390/math11214487

3. T. Antonova, R. Dmytryshyn, V. Kravtsiv, *Branched continued fraction expansions of Horn's hypergeometric function  $H_3$  ratios*, Mathematics, **9** (2021), 148. doi:10.3390/math9020148
4. T. Antonova, R. Dmytryshyn, I.-A. Lutsiv, S. Sharyn, *On some branched continued fraction expansions for Horn's hypergeometric function  $H_4(a, b, c, d; z_1, z_2)$  ratios*, Axioms, **12** (2023), 299. doi:10.3390/axioms12030299
5. T. Antonova, R. Dmytryshyn, S. Sharyn, *Generalized hypergeometric function  ${}_3F_2$  ratios and branched continued fraction expansions*, Axioms, **10** (2021), 310. doi:10.3390/axioms10040310
6. T. Antonova, R. Dmytryshyn, S. Sharyn, *Branched continued fraction representations of ratios of Horn's confluent function  $H_6$* , Constr. Math. Anal., **6** (2023), 22–37. doi:10.33205/cma.1243021
7. T.M. Antonova, R.I. Dmytryshyn, *Truncation error bounds for branched continued fraction  $\sum_{i_1=1}^N \frac{a_{i_1(1)}}{1} + \sum_{i_2=1}^{i_1} \frac{a_{i_2(2)}}{1} + \sum_{i_3=1}^{i_2} \frac{a_{i_3(3)}}{1} + \dots$* , Ukr. Math. J., **72** (2020), no.7, 1018–1029. doi:10.1007/s11253-020-01841-7
8. T.M. Antonova, R.I. Dmytryshyn, *Truncation error bounds for branched continued fraction whose partial denominators are equal to unity*, Mat. Stud., **54** (2020), no.1, 3–14. doi:10.30970/ms.54.1.3-14
9. T. Antonova, *On structure of branched continued fractions*, Carpathian Math. Publ., **16** (2024), 391–400. doi:10.15330/cmp.16.2.391-400
10. I.B. Bilanyk, D.I. Bodnar, O.G. Vozniak, *Convergence criteria of branched continued fractions*, Res. Math., **32** (2024), 53–69. doi:10.15421/242419
11. D.I. Bodnar, O.S. Bodnar, I.B. Bilanyk, *A truncation error bound for branched continued fractions of the special form on subsets of angular domains*, Carpathian Math. Publ., **15** (2023), 437–448. doi:10.15330/cmp.15.2.437-448
12. D.I. Bodnar, O.S. Bodnar, M.V. Dmytryshyn, M.M. Popov, M.V. Martsinkiv, O.B. Salamakha, *Research on the convergence of some types of functional branched continued fractions*, Carpathian Math. Publ., **16** (2024), 448–460. doi:10.15330/cmp.16.2.448-460
13. D.I. Bodnar, *Branched continued fractions*, Naukova Dumka, Kyiv, 1986. (in Russian)
14. D.I. Bodnar, *Expansion of a ratio of hypergeometric functions of two variables in branching continued fractions*, J. Math. Sci., **64** (1993), 1155–1158. doi:10.1007/BF01098839
15. M.G. Bustamante, J.E. Miraglia, F.D. Colavecchia, *Computation of a generalized Nordsieck integral*, Comput. Phys. Commun., **171** (2005), 40–48. doi:10.1016/j.cpc.2005.04.007
16. I. Chernega, R. Dmytryshyn, V. Kravtsiv, T. Vasylyshyn, A. Zagorodnyuk, *Function calculus on rings of multisets associated with symmetric and supersymmetric polynomials*, Carpathian Math. Publ., **17** (2025), no.1, 187–199. doi:10.15330/cmp.17.1.187-199
17. J. Choi, *Recent advances in special functions and their applications*, Symmetry, **15** (2023), 2159. doi:10.3390/sym15122159
18. M. Dmytryshyn, T. Goran, R. Rusyn, *Digital learning in public service*. In: BE. Boudriki Semlali, I. Ben Abdel Ouahab, F. Angeletti (eds) Technological Innovations for Sustainable Development, DATA 2025, Lecture Notes in Information Systems and Organisation, Springer, Cham, **81** (2025), 129–141. doi:10.1007/978-3-032-06725-8\_11
19. M. Dmytryshyn, V. Hladun, *On the sets of stability to perturbations of some continued fraction with applications*, Symmetry, **17** (2025), no.9, 1442. doi:10.3390/sym17091442
20. R. Dmytryshyn, T. Antonova, M. Dmytryshyn, *On the analytic extension of the Horn's confluent function  $H_6$  on domain in the space  $\mathbb{C}^2$* , Constr. Math. Anal., **7** (2024), 11–26. doi:10.33205/cma.1545452
21. R. Dmytryshyn, C. Cesarano, M. Dmytryshyn, I.-A. Lutsiv, *A priori bounds for truncation error of branched continued fraction expansions of Horn's hypergeometric functions  $H_4$  and their ratios*, Res. Math., **33** (2025), 13–22. doi:10.15421/242502
22. R. Dmytryshyn, C. Cesarano, I.-A. Lutsiv, M. Dmytryshyn, *Numerical stability of the branched continued fraction expansion of Horn's hypergeometric function  $H_4$* , Mat. Stud., **61** (2024), no.1, 51–60. doi:10.30970/ms.61.1.51-60
23. R. Dmytryshyn, V. Goran, *On the analytic extension of Lauricella–Saran's hypergeometric function  $F_K$  to symmetric domains*, Symmetry, **16** (2024), 220. doi:10.3390/sym16020220
24. R. Dmytryshyn, *On the analytic continuation of Appell's hypergeometric function  $F_2$  to some symmetric domains in the space  $\mathbb{C}^2$* , Symmetry, **16** (2024), 1480. doi:10.3390/sym16111480
25. R. Dmytryshyn, I. Nyzhnyk, *On approximation of some Lauricella–Saran's hypergeometric functions  $F_M$  and their ratios by branched continued fractions*, Mat. Stud., **62** (2025), no.2, 136–145. doi:10.30970/ms.62.2.136-145

26. R. Dmytryshyn, *Truncation error bounds for branched continued fraction expansions of some Appell's hypergeometric functions  $F_2$* , *Symmetry*, **17** (2025), 1204. doi:10.3390/sym17081204
27. N.P. Hoyenko, O.S. Manzij, *Expansion of Appel  $F_1$  and Lauricella  $F_D^{(N)}$  hypergeometric functions into branched continued fractions*, *Visnyk Lviv. Univ. Ser. Mech.-Math.*, **48** (1997), 17–26. (in Ukrainian)
28. V.R. Hladun, D.I. Bodnar, R.S. Rusyn, *Convergence sets and relative stability to perturbations of a branched continued fraction with positive elements*, *Carpathian Math. Publ.*, **16** (2024), 16–31. doi:10.15330/cmp.16.1.16-31
29. V.R. Hladun, M.V. Dmytryshyn, V.V. Kravtsiv, R.S. Rusyn, *Numerical stability of the branched continued fraction expansions of the ratios of Horn's confluent hypergeometric functions  $H_6$* , *Math. Model. Comput.*, **11** (2024), 1152–1166. doi:10.23939/mmc2024.04.1152
30. V.R. Hladun, N.P. Hoyenko, O.S. Manzij, L. Ventyk, *On convergence of function  $F_4(1, 2; 2, 2; z_1, z_2)$  expansion into a branched continued fraction*, *Math. Model. Comput.*, **9** (2022), 767–778. doi:10.23939/mmc2022.03.767
31. V. Hladun, V. Kravtsiv, M. Dmytryshyn, R. Rusyn, *On numerical stability of continued fractions*, *Mat. Stud.*, **62** (2024), 168–183. doi:10.30970/ms.62.2.168-183
32. V. Hladun, R. Rusyn, M. Dmytryshyn, *On the analytic extension of three ratios of Horn's confluent hypergeometric function  $H_7$* , *Res. Math.*, **32** (2024), 60–70. doi:10.15421/242405
33. V.R. Hladun, *Some sets of relative stability under perturbations of branched continued fractions with complex elements and a variable number of branches*, *Mat. method. and fiz.-mech. polya*, **57** (2014), 14–24. (in Ukrainian); Engl. transl.: *J. Math. Sci.*, **215** (2016), 11–25. doi:10.1007/s10958-016-2818-x
34. T.P. Hutchinson, *Compound gamma bivariate distributions*, *Metrika*, **28** (1981), 263–271. doi:10.1007/BF01902900
35. D. Kaliuzhnyi-Verbovetskyi, V. Pivovarchik, *Recovering the shape of a quantum caterpillar tree by two spectra*, *Mech. Math. Methods*, **5** (2023), no.1, 14–24. doi:10.31650/2618-0650-2023-5-1-14-24
36. B. Kol, R. Shir, *The propagator seagull: general evaluation of a two loop diagram*, *J. High Energy Phys.*, **2019** (2019), 83. doi:10.1007/JHEP03(2019)083
37. G. Lauricella, *Sulle funzioni ipergeometriche a più variabili*, *Rend. Circ. Matem.*, **7** (1893), 111–158. doi:10.1007/BF03012437
38. Y. Lutsiv, T. Antonova, R. Dmytryshyn, M. Dmytryshyn, *On the branched continued fraction expansions of the complete group of ratios of the generalized hypergeometric function  ${}_4F_3$* , *Res. Math.*, **32** (2024), 115–132. doi:10.15421/242423
39. M. Mazars, *Canonical partition functions of freely jointed chains*, *J. Phys. A Math. Gen.*, **31** (1998), 1949–1964. doi:10.1088/0305-4470/31/8/008
40. I. Nyzhnyk, R. Dmytryshyn, T. Antonova, *On branched continued fraction expansions of hypergeometric functions  $F_M$  and their ratios*, *Modern Math. Methods*, **3** (2025), 1–13. doi:10.64700/mmm.52
41. Sh. Saran, *Transformations of certain hypergeometric functions of three variables*, *Acta Math.*, **93** (1955), 293–312. doi:10.1007/BF02392525
42. O. Tulai, O. Motuzenko, O. Kvasnytsia, M. Dmytryshyn, L. Alekseyenko, N. Karpysyn, *International relations and women's leadership in financial management: from local to global socialization*, In: R. El Khoury (ed) *Strategic Decision-Making in Dynamic Business Environments. Studies in Systems, Decision and Control*, Springer, Cham, **642** (2026), 85–96. doi:10.1007/978-3-032-07220-7\_8

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Received 05.10.2025

Revised 30.01.2026