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LINEAR GEOMETRY: FLATS, RANKS, REGULARITY, PARALLELITY

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Linear geometry describes geometric properties that depend on the fundamental concept of a line. In this article, we survey the main concepts and results of linear geometry that depend on the flat hulls: flats, exchange, rank, regularity, modularity, and parallelity.

The article is the first in a series of reviews of linear geometry, the branch of geometry that studies geometric properties that depend only on lines, scheduled for publication in the journal *Matematychni Studii*. Linear geometry emerged as a separate branch of geometry in the 20th century. We present the main ideas and results of linear geometry using Bourbaki's approach to mathematics as a science that studies mathematical structures, in particular, such a fundamental mathematical structure of linear geometry as the structure of a liner.

1 INTRODUCTION

This paper is a first part of a series of surveys in Linear Geometry, an area of Geometry studying geometric properties that depend on lines only. Linear Geometry has been developed by many mathematicians since times of antiquity. It was formed as a separate area of Geometry in XX-th century. We present the main ideas and results of Linear Geometry using Bourbaki approach to mathematics as a science that studies mathematical structures. The fundamental mathematical structure of Linear Geometry is that of a liner.

Definition 1.1. A *liner* is a set X whose elements are called *points*, endowed with a family \mathcal{L} of subsets of X , called *lines*, that satisfy two axioms:

- (L1) any distinct points belong to a unique line;
- (L2) every line contains at least two distinct points.

The family of lines \mathcal{L} is called the *line structure* of the liner X .

This seemingly poor geometric structure allows to develop a rich geometric theory, full of fascinating theorems and intriguing open problems. A self-contained presentation of this theory can be found in the book [1] of the first author. In this paper we survey main notions and results of Chapters 1–6 of the book [1]. In those chapters we consider concepts that are expressible via flat hulls: ranks, modularity and parallelity.

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2 FLAT SETS IN LINERS

A subset A of a liner X is called *flat* if for any distinct points $x, y \in A$, the unique line \overline{xy} containing the points x, y is contained in the set A .

This definition implies that every subset $A \subseteq X$ of cardinality $|A| \leq 1$ is flat. The uniqueness of a line passing through two distinct points of a liner implies that every line in a liner is a flat set.

For a subset A of a liner X , the smallest flat set containing A is called the *flat hull* of A and is denoted by \overline{A} . The flat hull of the empty set is empty; the flat hull of a singleton is this singleton, and the flat hull of any doubleton $\{x, y\}$ is the line \overline{xy} passing through the points x, y .

Proposition 2.1. *For any subset A of a liner X , the flat hull \overline{A} is equal to the union $\bigcup_{n \in \omega} A_n$ of the sets defined by the recursive formula: $A_0 := A$ and $A_{n+1} = \bigcup_{x, y \in A_n} \overline{xy}$ for $n \in \omega$.*

Here we assume that $\overline{xy} = \{x\} = \{y\}$ if $x = y$.

Proposition 2.1 implies the following description of flat hulls in liners.

Corollary 2.2 ([1, 1.5.2]). *For any set A in a liner X , its flat hull \overline{A} equals $\bigcup_{F \in [A]^{<\omega}} \overline{F}$.*

Here $[A]^{<\omega}$ stands for the family of all finite subsets of a set A . For a set X we denote by $|X|$ the cardinality of X .

Definition 2.3. The *rank* $\|A\|$ of a subset A of a liner X is the smallest cardinality $|B|$ of a subset $B \subseteq X$ whose flat hull \overline{B} contains the set A . It is clear that the rank is monotone in the sense that $\|A\| \leq \|B\|$ for any sets $A \subseteq B$ in a liner.

A flat subset of rank 3 in a liner X is called a *plane* in X .

A subset A of a liner X is called a *hyperplane* in X if $A \neq X$ is a flat set such that $\overline{A \cup \{x\}} = X$ for all $x \in X \setminus A$.

Definition 2.4. Every subset A of a liner (X, \mathcal{L}) carries the induced line structure

$$\mathcal{L} \upharpoonright_A := \{L \cap A : L \in \mathcal{L} \wedge |L \cap A| \geq 2\}.$$

The liner $(A, \mathcal{L} \upharpoonright_A)$ is called a *subliner* of the liner (X, \mathcal{L}) .

3 EXCHANGE PROPERTIES IN LINERS

Definition 3.1. A liner X is defined to have the *Exchange Property* if for every flat $A \subseteq X$ and points $x \in X \setminus A$, $y \in \overline{A \cup \{x\}} \setminus A$ we have $x \in \overline{A \cup \{y\}}$.

The Exchange Property first appeared in the Steinitz Exchange Lemma, used by Steinitz for introducing the dimension of vector spaces.

The Exchange Property can be quantified as follows.

Definition 3.2. A liner X has a *κ -Exchange Property* for a cardinal κ if for any set $A \subseteq X$ of cardinality $|A| < \kappa$ and any points $x \in X$ and $y \in \overline{A \cup \{x\}} \setminus A$, we have $x \in \overline{A \cup \{y\}}$.

It is clear that a liner has the Exchange Property if and only if it has the κ -Exchange Property for every cardinal κ . Every liner has the κ -Exchange Property for all cardinals $\kappa \leq 2$.

Proposition 3.3 ([1, 2.1.6]). *A liner has the Exchange Property if and only if it has the n -Exchange Property for every $n \in \omega$.*

Proposition 3.4 ([1, 2.2.3]). *If a liner X has the κ -Exchange Property for some cardinal κ , then for every subset $A \subseteq X$ of rank $\|A\| \leq \kappa$, there exists a set $A' \subseteq A$ of cardinality $|A'| = \|A\|$ such that $A \subseteq \overline{A'}$.*

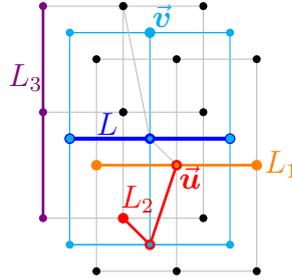
Example 3.5 (Terence Tao, see mathoverflow.net/a/450472/61536). There exists a liner that does not have the 3-Exchange Property.

Proof. Let $\mathbb{F}_3 = \{-1, 0, 1\}$ be a 3-element field, n, m be positive integers and $f: \mathbb{F}_3^n \rightarrow \mathbb{F}_3^m$ be an even function with $f(\{0\}^n) = \{0\}^m$. Endow the set $X = \mathbb{F}_3^n \times \mathbb{F}_3^m$ with the family of lines

$$\mathcal{L} = \left\{ \{(x, x'), (y, y'), (z, z')\} \in [X]^3 : x + y + z = 0 \wedge x' + y' + z' = f(x - z) \right\}.$$

Here $[X]^3 = \{(x, y, z) \in X^3 : |\{x, y, z\}| = 3\}$. It can be shown that X is a liner. Now we select a special even function f for which this liner does not have 3-Exchange Property. For $n = 2$ and $m = 1$, consider the even function $f: \mathbb{F}_3^2 \rightarrow \{0, 1\} \subseteq \mathbb{F}_3$ such that $f^{-1}(1) = \{(0, 1), (0, -1)\}$. For such a function f , the set $L = \{(x, 0, 0) : x \in \mathbb{F}_3\}$ is a line in (X, \mathcal{L}) such that for the point $\vec{u} = (0, 1, 0)$ the flat hull $\overline{L \cup \{\vec{u}\}}$ coincides with X . On the other hand, for the point $\vec{v} = (0, 0, 1) \in \overline{L \cup \{\vec{u}\}} \setminus L$ the flat hull $\overline{L \cup \{\vec{v}\}}$ coincides with the set $\mathbb{F}_3 \times \{0\} \times \mathbb{F}_3$, which is strictly smaller than $\overline{L \cup \{\vec{u}\}} = X$. This means that the liner X does not have the 3-Exchange Property.

Observe that $L_1 := \{(-1, 1, 0), (0, 1, 0), (1, 1, 0)\}$ and $L_2 := \{(0, -1, -1), (0, 0, -1), (0, 1, 0)\}$ are two distinct lines in the plane $X = \overline{L \cup \{\vec{u}\}}$ that contain the point \vec{u} and are disjoint with the line L . Also the lines L_2 and $L_3 := \{(-1, -1, -1), (-1, -1, 0), (-1, -1, 1)\}$ differ by the following properties: $\forall x \in L_3 (L_3 \subseteq \overline{L_2 \cup \{x\}})$ but $\forall x \in L_2 (L_2 \not\subseteq \overline{L_3 \cup \{x\}})$.



□

4 RANKED LINERS

Definition 4.1. A liner X is called *ranked* if any two flats $A \subseteq B \subseteq X$ of the same finite rank $\|A\| = \|B\| < \omega$ are equal.

Remark 4.2. A liner X is ranked if and only if its rank function $\|\cdot\|$ is strictly monotone in the sense that distinct flats $A \subset B$ of finite rank have distinct ranks $\|A\| < \|B\|$.

Definition 4.1 can be quantified as follows.

Definition 4.3. A liner X is called κ -ranked for a cardinal κ if any two flats $A \subseteq B \subseteq X$ of finite rank $\|A\| = \|B\| \leq \kappa$ are equal.

It is clear that a liner is ranked if and only if it is κ -ranked for every cardinal κ if and only if it is n -ranked for every $n \in \omega$. Every liner is 2-ranked.

The following theorem shows that the rankedness of liners is equivalent to the Exchange Property.

Theorem 4.4 ([1, 2.3.5]). *Let κ be a cardinal. A liner is κ -ranked if and only if it has the κ -Exchange Property.*

Corollary 4.5. *A liner is ranked if and only if it has the Exchange Property.*

The notion of a ranked liner allows us to prove the following characterization of planes in 3-ranked liners.

Proposition 4.6 ([1, 2.3.7]). *Let X be a 3-ranked liner.*

- (1) *A set $A \subseteq X$ is a plane if and only if $A = \overline{\{x, y, z\}}$ for some points $x \in X$, $y \in X \setminus \{x\}$, and $z \in X \setminus \overline{xy}$.*
- (2) *If a set $A \subseteq X$ is a plane, then $A = \overline{\{x, y, z\}}$ for every points $x \in A$, $y \in A \setminus \{x\}$, and $z \in X \setminus \overline{xy}$.*

Example 4.7. The liner in Example 3.5 is not 3-ranked: it is a plane containing a proper subplane.

5 THE RELATIVE RANK AND CODIMENSION OF SETS IN LINERS

Definition 5.1. For two sets A, B in a liner X , let $\|A\|_B$ be the smallest cardinality of a set $C \subseteq X$ such that $A \subseteq \overline{B \cup C}$. The cardinal $\|A\|_B$ is called the *B-relative rank* (or just the *B-rank*) of the set A in X . If $B \subseteq A$, then the *B-rank* $\|A\|_B$ is denoted by $\dim_B(A)$ and called the *codimension* of B in A .

Observe that every set A in a liner has rank $\|A\| = \|A\|_\emptyset$. So, the rank is a special case of the *B-rank* for $B = \emptyset$.

It is easy to see that the relative rank is bimonotone in the sense that $\|A\|_C \leq \|B\|_D$ for any sets $A \subseteq B$ and $D \subseteq C$ in a liner.

Proposition 5.2 ([1, 2.4.3]). *If a liner X has the κ -Exchange Property for some cardinal κ , then for every subsets $A, B \subseteq X$ with $\|A \cup B\| \leq \kappa$, there exist sets $A' \subseteq A$ and $B' \subseteq B \setminus \overline{A}$ such that $A \subseteq \overline{A'}$, $B \subseteq \overline{A' \cup B'}$, $|A' \cup B'| = \|A \cup B\|$, and $|B'| = \|B\|_A$.*

6 INDEPENDENCE IN LINERS

Definition 6.1. A subset A of a liner X is *independent* if $a \notin \overline{A \setminus \{a\}}$ for every $a \in A$.

It is easy to see that any subset of an independent set in a liner is independent.

The notion of an independent set is a partial case of a more general notion of a *B-independent* set.

Definition 6.2. Let B be a set in a liner X . A set $A \subseteq X$ is called *B-independent* if $a \notin \overline{B \cup (A \setminus \{a\})}$ for every $a \in A$.

Therefore, a set A in a liner X is independent if and only if it is \emptyset -independent.

Proposition 6.3 ([1, 2.5.8]). *Let A be a set in a liner X , I be an A -independent set in X and J is an $(A \cup I)$ -independent set in X . If the liner X has the $\|A \cup I \cup J\|$ -Exchange Property, then the set $I \cup J$ is A -independent.*

Proposition 6.4 ([1, 2.5.9]). *Let A, I be two sets in a liner X . If the set I is A -independent and the liner X is $\|A \cup I\|$ -ranked, then for every point $x \in X \setminus \overline{A \cup I}$, the set $I \cup \{x\}$ is A -independent.*

Definition 6.5. Let B, A be two sets in a liner X . A B -independent set $I \subseteq A$ is called a *maximal B -independent in A* if $I = J$ for any B -independent set J with $I \subseteq J \subseteq A$.

The Kuratowski–Zorn Lemma implies that for every sets A, B in a liner X , every B -independent set $I \subseteq A$ can be enlarged to a maximal B -independent set $J \subseteq A$.

Theorem 6.6 ([1, 2.5.14]). *Let A, B be any sets in a liner X . If the liner X has the $\|A \cup B\|$ -Exchange Property, then the A -rank $\|B\|_A$ of the set B is equal to the cardinality of any maximal A -independent set in B .*

Theorem 6.6 implies the following corollary.

Corollary 6.7 ([1, 2.5.15]). *Let A be a set in a liner X . If the liner X has the $\|A\|$ -Exchange Property, then the rank $\|A\|$ of the set A is equal to the cardinality of any maximal independent set in A .*

Proposition 6.8 ([1, 2.5.16]). *For every sets A, B, C in a liner X ,*

$$\|C\|_A \leq \|B\|_A + \|C\|_B.$$

If $\overline{A} \subseteq \overline{B} \subseteq \overline{C}$ and the liner X has the $\|C\|$ -Exchange Property, then $\|C\|_A = \|B\|_A + \|C\|_B$.

Corollary 6.9 ([1, 2.5.17]). *For every sets A, B in a liner X , $\|B\| \leq \|A\| + \|B\|_A$. If $\overline{A} \subseteq \overline{B}$ and the liner X has the $\|B\|$ -Exchange Property, then $\|B\| = \|A\| + \|B\|_A$.*

7 REGULARITY AXIOMS

In this section we introduce three properties of liners, describing the structure of the hull $\overline{A \cup \{a\}}$ of a flat A with an attached singleton $\{a\}$.

For two subsets A, B of a liner X and a point $x \in X$, let

$$\overline{Ax} = \overline{x\overline{A}} := \bigcup_{a \in A} \overline{ax} \quad \text{and} \quad \overline{A\overline{B}} := \bigcup_{x \in A} \bigcup_{y \in B} \overline{xy} = \bigcup_{a \in A} \overline{a\overline{B}} = \bigcup_{b \in B} \overline{A\overline{b}}.$$

Definition 7.1. A liner X is called

- *strongly regular* if for every nonempty flat $A \subseteq X$ and point $b \in X \setminus A$, we have $\overline{A \cup \{b\}} = \overline{A\overline{b}}$;
- *regular* if for every flat $A \subseteq X$ and points $a \in A, b \in X \setminus A$ we have $\overline{\overline{A \cup \{b\}}} = \bigcup_{y \in \overline{a\overline{b}}} \overline{Ay}$;
- *proregular* if for every flat $A \subseteq X$ and points $a \in A, b \in X \setminus A$ with $\overline{a\overline{b}} \neq \{a, b\}$, we have $\overline{A \cup \{b\}} = \bigcup_{y \in \overline{a\overline{b}}} \overline{Ay}$;
- *weakly regular* if for every flat $A \subseteq X$ and points $a \in A, b \in X \setminus A$ we have $\overline{\overline{A \cup \{b\}}} = \bigcup_{x \in A} \overline{\{a, b, x\}}$.

Each strongly regular liner is regular and each regular liner is proregular and weakly regular. The notion of a (pro)regular liner can be quantified as follows.

Definition 7.2. Let κ be a cardinal. A liner X is called κ -regular (resp. κ -proregular) if for every set $A \subseteq X$ of cardinality $|A| < \kappa$ and every points $o \in \overline{A}, p \in X \setminus \overline{A}$ (such that $\overline{o\overline{p}} \neq \{o, p\}$), we have $\overline{\{p\} \cup A} = \bigcup_{u \in \overline{o\overline{p}}} \bigcup_{a \in \overline{A}} \overline{ua}$.

A liner is (pro)regular if and only if it is κ -(pro)regular for every cardinal κ .

Proposition 7.3 ([1, 3.1.6]). *Let κ be a cardinal. Every κ -proregular liner is κ -ranked and has the κ -Exchange Property.*

Example 7.4. The 13-element group \mathbb{Z}_{13} endowed with family of lines

$$\mathcal{L} := \{a + \{0, 3, 4\}, a + \{0, 5, 7\} : a \in \mathbb{Z}_{13}\}$$

is an example of a finite liner, which is ranked but not 3-proregular.

Proposition 7.3 implies the following corollary.

Corollary 7.5 ([1, 3.1.8]). *Every proregular liner is ranked and has the Exchange Property.*

The Exchange Property in weakly regular liners is equivalent to the 3-Exchange Property.

Proposition 7.6 ([1, 3.1.9]). *A weakly regular liner X is ranked if and only if it is 3-ranked.*

Theorem 7.7 ([1, 3.1.10+3.1.11]). *A liner X is regular if and only if it is 4-regular if and only if it is weakly regular and 3-regular.*

8 MODULAR AND WEAKLY MODULAR LINERS

Theorem 8.1 ([1, 5.1.1]). *For any flats A, B in a ranked liner X , we have the inequality $\|A \cap B\| + \|A \cup B\| \leq \|A\| + \|B\|$. If $\|A \cup B\|$ is infinite, then $\|A \cap B\| + \|A \cup B\| = \|A\| + \|B\|$.*

Definition 8.2. A liner X is called (weakly) modular if $\|A \cap B\| + \|A \cup B\| = \|A\| + \|B\|$ for any flats $A, B \subseteq X$ (with $A \cap B \neq \emptyset$).

Theorem 8.3 ([1, 5.3.2]). *For a ranked liner X the following conditions are equivalent:*

- (1) X is (weakly) modular;
- (2) $A \cap \overline{B \cup C} = \overline{(A \cap B) \cup C}$ for any flats $A, B \subseteq X$ and set $C \subseteq A$ (with $A \cap B \neq \emptyset$);
- (3) $A \cap \overline{B \cup C} = \overline{(A \cap B) \cup C}$ for any flats $A, B \subseteq X$ and set $C \subseteq A$ with $\|A \cup B\| < \omega$ (and $A \cap B \neq \emptyset$).

Two lines A, B in a liner are *coplanar* if they are contained in some plane.

Theorem 8.4 ([1, 5.5.1]). *A liner X is modular if and only if it is strongly regular if and only if X contains no disjoint coplanar lines.*

Theorem 8.5 ([1, 5.4.1]). *For a liner X the following conditions are equivalent:*

- (1) X is weakly modular;
- (2) X is ranked and weakly regular;
- (3) X is ranked and for every planes $P \cup \Pi$ with $\|P \cup \Pi\| = 4$, the intersection $P \cap \Pi$ is not a singleton.

Corollary 8.6. *Every regular liner is weakly modular.*

Example 8.7. The Euclidean plane is an example of a regular weakly modular liner which is not strongly regular and hence not modular.

9 PARALLELITY POSTULATES AND AXIOMS

In this section we introduce many parallelity postulates and axioms that will be studied in the next sections. We start with four classical Parallelity Postulates.

Definition 9.1. A liner X is defined to be

- *Proclus* if for every plane $P \subseteq X$, line $L \subseteq P$, and point $x \in P \setminus L$ there exists at most one line Λ in X such that $x \in \Lambda \subseteq P \setminus L$;
- *Playfair* if for every plane $P \subseteq X$, line $L \subseteq P$, and point $x \in P \setminus L$ there exists a unique line Λ in X such that $x \in \Lambda \subseteq P \setminus L$;

- *Bolyai* if for every plane $P \subseteq X$, line $L \subseteq P$, and point $x \in P \setminus L$ there exists at least one line Λ in X such that $x \in \Lambda \subseteq P \setminus L$;
- *Lobachevsky* if for every plane $P \subseteq X$, line $L \subseteq P$, and point $x \in P \setminus L$ there exist at least two distinct lines Λ in X such that $x \in \Lambda \subseteq P \setminus L$.

These four definitions are partial cases of the following Parallel Postulates involving a cardinal parameter κ .

Definition 9.2. Let κ be a cardinal. A liner X is defined to be

- κ -*hypoparallel* if for every plane $P \subseteq X$, line $L \subseteq P$, and point $x \in P \setminus L$ there exist at most κ lines Λ such that $x \in \Lambda \subseteq P \setminus L$;
- κ -*parallel* if for every plane $P \subseteq X$, line $L \subseteq P$, and point $x \in P \setminus L$ there exist exactly κ lines Λ such that $x \in \Lambda \subseteq P \setminus L$;
- κ -*hyperparallel* if for every plane $P \subseteq X$, line $L \subseteq X$, and point $x \in P \setminus L$ there exist at least κ lines Λ such that $x \in \Lambda \subseteq P \setminus L$.

It is clear that a liner is κ -parallel if and only if it is both κ -hypoparallel and κ -hyperparallel.

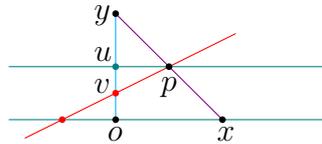
Observe that a line X is Proclus, Playfair, Bolyai, Lobachevsky if and only if it is 1-hypoparallel, 1-parallel, 1-hyperparallel, 2-hyperparallel, respectively.

In Theorems 13.1 and 14.2, we shall show that Proclus and Playfair Parallel Postulates are first-order properties of liners, whose first-order characterizations involve the following parallelity axioms.

Definition 9.3. A liner X is defined to be

- *projective* if $\forall o, x, y \in X \forall p \in \overline{xy} \forall v \in \overline{oy} \setminus \{p\} (\overline{vp} \cap \overline{ox} \neq \emptyset)$;
- *proaffine* if $\forall o, x, y \in X \forall p \in \overline{xy} \setminus \overline{ox} \exists u \in \overline{oy} \forall v \in \overline{oy} \setminus \{u\} (\overline{vp} \cap \overline{ox} \neq \emptyset)$;
- *affine* if $\forall o, x, y \in X \forall p \in \overline{xy} \setminus \overline{ox} \exists u \in \overline{oy} \forall v \in \overline{oy} (u = v \Leftrightarrow \overline{vp} \cap \overline{ox} = \emptyset)$;
- *hyperaffine* if $\forall o, x, y \in X \forall p \in \overline{xy} \setminus \overline{ox} \exists u \in \overline{oy} (\overline{up} \cap \overline{ox} = \emptyset)$;
- *hyperbolic* if $\forall o, x, y \in X \forall p \in \overline{xy} \setminus (\overline{ox} \cup \overline{oy}) \exists u, v \in \overline{oy} (u \neq v \wedge \overline{up} \cap \overline{ox} = \emptyset = \overline{vp} \cap \overline{ox})$;
- *injective* if $\forall o, x, y \in X \forall p \in \overline{xy} \setminus (\overline{ox} \cup \overline{oy}) \forall v \in \overline{oy} \setminus \{o, y\} (\overline{vp} \cap \overline{ox} = \emptyset)$.

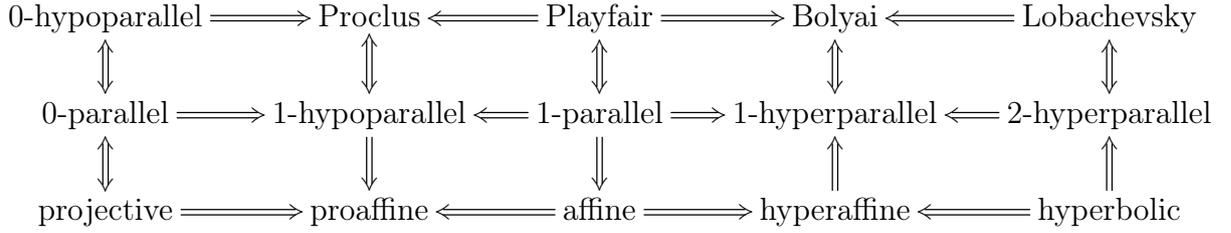
The first-order formulas defining projective, proaffine, affine, hyperaffine, and hyperbolic liners will be called the *projectivity*, *proaffinity*, *affinity*, *hyperaffinity*, *hyperbolicity*, and *injectivity axioms*, respectively.



Remark 9.4. The proaffinity, affinity, hyperaffinity, and hyperbolicity axioms are first order counterparts of the parallel postulates of Proclus, Playfair, Bolyai, and Lobachevsky, respectively. The projectivity axiom has been explicitly formulated by Oswald Veblen in his axiom system for Projective Geometry. The injectivity axiom (as an antonym to the projectivity) is well-known in the theory of unitals, as the absence of O’Nan configurations (known also as Pasch configurations), see Section 16 for more information on the classical unitals.

Remark 9.5. Every 3-long injective liner is hyperaffine, and every 4-long injective liner is hyperbolic. A liner X is κ -long if every line L in X has cardinality $|L| \geq \kappa$.

For 3-long liners, the above parallelity postulates and axioms relate as follows.



Proposition 9.6 ([1, 3.2.8]). *A liner X is injective if and only if among any four distinct lines in X two are disjoint.*

Example 9.7. For every $n \geq 2$, the subliner $B = \{x \in \mathbb{R}^n : \sum_{i \in n} x_i^2 < 1\}$ of the Euclidean space \mathbb{R}^n is hyperbolic. This hyperbolic liner is called the *Beltrami–Klein model* of hyperbolic geometry.

10 BALANCED LINERS

Given a cardinal κ , we say that a liner X is κ -balanced if all flats of rank κ have the same cardinality. This cardinality is denoted by $|X|_\kappa$. A liner X is *balanced* if it is κ -balanced for all cardinals κ .

For a κ -balanced liner X , the cardinal $|X|_\kappa$ is well-defined if X contains a flat $F \subseteq X$ of rank $\|F\| = \kappa$. In this case $|X|_\kappa = |F|$ for every flat $F \subseteq X$ of rank $\|F\| = \kappa$.

It is clear that every liner X is 1-balanced with $|X|_1 = 1$. A liner X is 2-balanced if and only if all lines have the same cardinality, denoted by $|X|_2$.

Theorem 10.1 ([1, 3.3.3]). *Let X be a 2-balanced liner and \mathcal{L} be the family of all lines in X . If $|X|_2 < |X|$, then for every point $x \in X$, the cardinality of the family $\mathcal{L}_x := \{L \in \mathcal{L} : x \in L\}$ satisfies two equations $|X| - 1 = (|X|_2 - 1) \cdot |\mathcal{L}_x|$ and $|X|_2 \cdot |\mathcal{L}| = |X| \cdot |\mathcal{L}_x|$, and hence $|\mathcal{L}_x|$ does not depend on x . If X is finite, then $|X|_2$ divides the number $(|\mathcal{L}_x| - 1) \cdot |\mathcal{L}_x|$.*

Remark 10.2. Theorem 10.1 implies that the cardinality $v := |X|$ of any finite 2-balanced liner X with $k := |X|_2 < |X|$ satisfies the equality $v = 1 + (k - 1) \cdot r$ for some number $r \geq k$ such that k divides $(r - 1) \cdot r$ (denoted by $k \mid (r - 1)r$). The latter divisibility condition implies (and in fact, is equivalent to) the divisibilities $(k - 1) \mid (v - 1)$ and $(k - 1)k \mid (v - 1)v$. Pairs of positive integer numbers (k, v) satisfying the divisibility conditions

$$(k - 1) \mid (v - 1) \quad \text{and} \quad (k - 1)k \mid (v - 1)v$$

are called *admissible*. Therefore, for every finite 2-balanced liner X , the pair $(|X|_2, |X|)$ is admissible. The admissibility of a pair (k, v) does not imply the existence of a 2-balanced liner X with $(|X|_2, |X|) = (k, v)$. For example, the pairs $(6, 36)$ and $(7, 43)$ are admissible, but liners X with $(|X|_2, |X|) \in \{(6, 36), (7, 43)\}$ do not exist, by the Bruck–Ryser’s Theorem 10.20. The pair $(6, 46)$ is admissible but no liner X with $(|X|_2, |X|) = (6, 46)$ exists, by a result of Houghten, Thiel, Janssen, and Lam [15], proved by a computer search.

On the other hand, the admissibility condition is asymptotically sufficient for the existence of finite 2-balanced liners, according to the following fundamental theorem, proved by Wilson in [20], [21], [22].

Theorem 10.3 (Wilson, 1975). *For every integer numbers k there exists a number v_k such that for every admissible pair (k, v) with $v > v_k$, there exists a 2-balanced liner X such that $|X|_2 = k$ and $|X| = v$.*

Remark 10.4. We can assume that the number v_k in Wilson's Theorem 10.3 is the smallest possible. In this case it is determined uniquely. The Wilson proof of Theorem 10.3 does not provide an explicit formula for finding the number v_k , it just claims that it does exist. However, for $k \leq 9$ the following upper and lower bounds for the numbers v_k are known (see [8, §II.3.1]):

k	2	3	4	5	6	7	8	9
$v_k \geq$	2	3	4	5	46	43	8	9
$v_k \leq$	2	3	4	5	801	2605	3753	16497

As we have already mentioned, liners X with $|X|_2 = 6$ and $X \in \{36, 46\}$ do not exist. The existence of liners X with $|X|_2 = 6$ is undecided for the following values v (see Table 3.4 in [8, II.3.1] and [2]): 51, 61, 81, 166, 231, 256, 261, 286, 316, 321, 346, 351, 376, 406, 411, 436, 471, 501, 561, 591, 616, 646, 651, 676, 771, 796, 801.

Theorem 10.5 ([1, 3.3.7]). *If a liner X is 2-balanced and κ -parallel for some cardinal κ , then X is 3-balanced with $|X|_3 = 1 + (\kappa + |X|_2)(|X|_2 - 1)$. If the cardinal $|X|_3$ is well-defined and finite, then the liner X is 3-ranked.*

Proposition 10.6 ([1, 3.3.8]). *If a liner X of rank $\|X\| \geq 3$ is 2-balanced and 3-balanced, then there exists a unique cardinal $\lambda \geq |X|_2$ such that $|X|_3 - 1 = \lambda \cdot (|X|_2 - 1)$. If $|X|_2 < |X|_3$, then X is κ -parallel for the unique cardinal κ such that $\lambda = \kappa + |X|_2$.*

Theorem 10.7 ([1, 3.3.9]). *Assume that a liner X is κ -parallel for some finite cardinal κ . If $\kappa > 0$ or X is 3-long, then X is 2-balanced, 3-balanced and $|X|_3 = 1 + (|X|_2 - 1)(\kappa + |X|_2)$. If $|X|_2$ is finite, then $|X|_2$ divides $(\kappa - 1)\kappa$.*

Theorems 10.5 and Proposition 10.6 imply the following characterization.

Corollary 10.8 ([1, 3.3.11]). *For a 3-long finite liner X of rank $\|X\| \geq 3$, the following conditions are equivalent:*

- (1) X is κ -parallel for some cardinal κ ;
- (2) X is 2-balanced and 3-balanced;
- (3) X is 2-balanced, 3-balanced, and κ -parallel for a unique cardinal κ satisfying the equation $|X|_3 - 1 = (\kappa + |X|_2)(|X|_2 - 1)$.

Proposition 10.9 ([1, 3.3.13]). *If a liner X is 2-balanced, then for every infinite cardinal κ , the liner X is κ -balanced with $|X|_\kappa = \max\{\kappa, |X|_2\}$.*

Theorem 10.10 ([1, 3.3.14]). *If a weakly regular 3-ranked liner X is 2-balanced and κ -parallel for some cardinal κ , then X is balanced with $|X|_n = 1 + (|X|_2 - 1) \sum_{r=0}^{n-2} (\kappa + |X|_2 - 1)^r$ for every finite cardinal $n \geq 2$.*

Question 10.11. *For which cardinals κ there exist weakly regular 3-long κ -balanced ranked liners of rank $\|X\| > 3$?*

Proposition 10.12 ([1, 3.3.16]). *Every 2-balanced liner X with*

$$|X|_2 < |X| < |X|_2^3 - 2 \cdot |X|_2^2 + 2 \cdot |X|_2$$

is a ranked plane.

Theorem 10.13 ([1, 3.3.17]). *Every affine liner X is 2-balanced.*

Remark 10.14. By Theorem 10.13, all lines in an affine liner have the same cardinality, which is called the *order* of the affine liner. By the Veblen–Young Theorem (1908), the order of any finite affine regular liner of rank ≥ 4 is a power of a prime number. There is a (still

unproved) conjecture that the same result is true for finite affine planes: their orders are powers of primes.

Proposition 10.15 ([1, 3.3.24]). *Every 3-long projective liner X is 2-balanced.*

If a liner X of rank $\|X\| \geq 2$ is not 3-long, then it contains a line of length 2 and there is no further restrictions on the length of lines in such liners.

Example 10.16. For every set K of non-zero cardinals with $1, 2 \in K$, there exists a projective liner X of cardinality $|X| = \sum_{k \in K} k$ and rank $2 \cdot |K| - 1$ such that $K = \{\overline{xy} : x, y \in X\}$.

Corollary 10.17 ([1, 3.3.26+8.5.2]). *Any lines L, Λ in a 3-long proaffine (and Proclus) liner X have $|L| \leq |\Lambda| + 2$ (and $|L| \leq |\Lambda| + 1$).*

Corollary 10.18 ([1, 3.3.22]). *Every ω -long proaffine liner is 2-balanced.*

Remark 10.19. If all lines in a projective liner have the same finite cardinality λ , then the number $\lambda - 1$ is called the *order* of the projective liner. By the Veblen–Young Theorem, the order of a finite projective liner of rank ≥ 4 is a power of a prime number. There is a (still unproved) conjecture that the same result holds for projective planes: their order is a power of a prime number. Using projective completions (discussed in Chapter 7 of [1]), it can be shown that a projective plane of order n exists if and only if an affine plane of order n exists.

The only available classical result on possible orders of finite affine planes is the following fundamental theorem of Bruck–Ryser [6].

Theorem 10.20 (Bruck–Ryser, 1949; [1, 25.5.1]). *If a number $n \in (4\mathbb{Z} + 1) \cup (4\mathbb{Z} + 2)$ is an order of an affine or projective plane, then $n = x^2 + y^2$ for some integer numbers x, y .*

Bruck–Ryser Theorem implies that the order of an affine or projective plane cannot be equal to 6, 14, or 22. On the other hand, the number $10 = 4 \cdot 2 + 2 = 1^2 + 9^2$ is the sum of two squares but no affine plane of order 10 exists, as was shown by Lam, Thiel and Swiercz [16] in 1989, using heavy computer calculations. The problem of the (non-)existence of an affine or projective plane of order 12 remains open, out of reach of modern (super)computers.

11 THE REGULARITY OF BALANCED LINERS

In this section we discuss the structure of weakly modular balanced liners.

Lemma 11.1 ([1, 5.6.1]). *Let X be a weakly modular liner of rank $\|X\| \geq 4$. If some plane $P \subseteq X$ contains two disjoint lines, then P is the union of a family of pairwise disjoint lines.*

Corollary 11.2 ([1, 5.6.3]). *Let X be a balanced weakly modular liner of rank $\|X\| \geq 4$. If X is not projective, then $|X|_3 = |X|_2 \cdot \kappa$ for some cardinal $\kappa \geq |X|_2$.*

Lemma 11.3 ([1, 5.6.4]). *Let X be a 2-balanced and 3-balanced weakly modular liner. If $|X|_3 < \omega$ and X is not projective, then for every flat $Y \subseteq X$ of rank $\|Y\| = 4$, every plane $P \subseteq Y$ and every point $y \in Y \setminus P$, there exists a plane $\Pi \subseteq Y$ such that $y \in \Pi \subseteq Y \setminus P$.*

The following lemma is a higher-dimensional counterpart of Lemma 11.1.

Lemma 11.4 ([1, 5.6.5]). *Let X be a weakly modular liner of rank $\|X\| \geq 5$. If some flat $Y \subseteq X$ of rank $\|Y\| = 4$ contains two disjoint planes, then Y is the union of a family of pairwise disjoint planes.*

This lemma implies the following higher-dimensional counterpart of Corollary 11.2.

Corollary 11.5 ([1, 5.6.7]). *Let X be a balanced weakly modular liner of rank $\|X\| \geq 5$. If X is not projective, then $|X|_4 = |X|_3 \cdot \kappa$ for some cardinal $\kappa \geq |X|_2$.*

The principal results of this section are the following theorems of Jean Doyen and Xavier Hubault ([10]).

Theorem 11.6 (Doyen-Hubaut, 1971; [1, 5.6.8+5.6.10]). *Let X be a balanced weakly modular liner of rank $\|X\| \geq 4$. If $|X|_3 < \omega$, then X is p -parallel for some*

$$p \in \{0, 1, |X|_2^2 - |X|_2 + 1, |X|_2^3 + 1\}.$$

If $\|X\| \geq 5$, then $p \in \{0, 1\}$ and hence X is proaffine.

Problem 11.7. *Is every finite balanced weakly modular liner of rank ≥ 4 proaffine?*

Theorem 11.8 ([1, 5.6.12]). *For a balanced liner X with $\|X\| \geq 4$ and $|X|_3 < \omega$, the following conditions are equivalent:*

- (1) X is regular; (2) X is p -parallel for some $p \in \{0, 1\}$; (3) X is projective or Playfair;
- (4) X is Proclus; (5) X is 3-proregular and proaffine.

If $\|X\| \geq 5$, then the conditions (1)–(5) are equivalent to:

- (6) X is weakly regular; (7) X is weakly modular.

Corollary 11.9 ([1, 5.6.13]). *For a finite liner X of rank $\|X\| \geq 5$, the following conditions are equivalent:*

- (1) X is regular and Playfair; (2) X is balanced, weakly regular and not strongly regular;
- (3) X is balanced, weakly modular and not modular.

Problem 11.10. *Is every finite balanced finite affine liner X of rank 4 regular?*

Remark 11.11. By computer calculation, Ivan Hetman has established that every balanced regular liner X with $|X|_2 \leq 4$ is proaffine.

Example 11.12. There exists a hyperbolic (and hence not proaffine) regular balanced liner X with $|X|_2 = 8$ and $|X| = |X|_3 = 120$. This liner is one of maximal arcs, discovered by Denniston [9].

12 PROJECTIVE LINERS

In this section we present some characterizations of projective liners.

Two lines A, B in a liner X are *skew* if $\|A \cup B\| = 4$. It is clear that any skew lines are disjoint. The converse is true in projective liners.

Theorem 12.1 ([1, 3.4.1+5.5.1]). *For every liner X , the following conditions are equivalent:*

- (1) X is 0-hypoparallel; (2) X is 0-parallel; (3) any disjoint lines in X are skew;
- (4) X projective; (5) X is strongly regular; (6) X is modular.

Corollary 12.2 ([1, 3.4.3]). *A 2-balanced liner X with $|X|_2 < \omega$ is projective if and only if X is 3-balanced and $|X|_3 = |X|_2^2 - |X|_2 + 1$.*

Theorems 12.1 and 10.10 imply the following corollary.

Corollary 12.3 ([1, 3.4.4]). *Every finite projective liner X of order ℓ contains exactly $\sum_{k=0}^{\|X\|-1} \ell^k = \frac{\ell^{\|X\|} - 1}{\ell - 1}$ points.*

By Proposition 10.15, every 3-long projective liner is 2-balanced. Now we describe the structure of projective liners which are not 3-long. This description involves the notion of a maximal n -long flat in a liner.

Definition 12.4. Let κ be a cardinal number. A flat B in a liner X is called κ -long if $|\overline{xy}| \geq \kappa$ for every distinct points $x, y \in B$. A κ -long flat B in X is called *maximal κ -long* if every κ -long flat $A \subseteq X$ containing B coincides with B .

Remark 12.5. By Definition 12.4, every flat $F \subseteq X$ of cardinality $|F| \leq 1$ in a liner X is κ -long for every cardinal κ . The Kuratowski-Zorn Lemma implies that every κ -long flat in a liner X is a subset of a maximal κ -long flat in X . In particular, every point x of a liner X belongs to some maximal κ -long flat (which can be equal to the singleton $\{x\}$, if the cardinal κ is too large).

A family of sets \mathcal{F} is called *disjoint* if $A \cap B = \emptyset$ for every distinct sets $A, B \in \mathcal{F}$. The following theorem shows that the line structure of a projective liner is uniquely determined by the structures of its maximal 3-long flats.

Theorem 12.6 ([1, 3.4.11]). *The family \mathcal{M} of maximal 3-long flats in a projective liner X is disjoint and $\overline{ab} = \{a, b\}$ for any distinct flats $A, B \in \mathcal{M}$ and points $a \in A$ and $b \in B$.*

Next, we describe the structure of projective planes which are not 3-long.

Definition 12.7. A liner X is called a *near-pencil* if X contains a line $L \subseteq X$ such that $X \setminus L$ is a singleton.

In the following theorem by a *projective plane* we understand a projective liner of rank 3.

Theorem 12.8 ([1, 3.4.13]). *A liner is a near-pencil if and only if it is projective plane which is not 3-long.*

Corollary 12.9 ([1, 3.4.16]). *Every line in a projective liner X has nonempty intersection with every hyperplane in X .*

We recall that a flat H in a liner X is a *hyperplane* if $H = X$ and every flat F in X with $H \subseteq F \subseteq X$ is equal to H or X .

Theorem 12.10 (Bruck, 1955; [1, 3.5.1]). *Let X be a projective liner of finite order n and P be a projective subliner of order $p < n$ in X . If $\|P\| \geq 3$, then either $n = p^2$ or else $n \geq p^2 + p$.*

Theorem 12.10 motivates the following long-standing open problem.

Problem 12.11. *Is there a projective plane X of composite order $p^2 + p$ for some p ? In particular, is there a projective plane of order $12 = 3^2 + 3$?*

13 PROAFFINE AND PROCLUS LINERS

In this section we study the interplay between proaffine and Proclus liners, and will establish that Proclus liners are exactly proaffine 3-proregular liners.

Theorem 13.1 ([1, 3.6.1]). *For a liner X , the following conditions are equivalent:*

- (1) X is Proclus;
- (2) X is proaffine and 3-proregular;
- (3) for every line $L \subseteq X$ and points $o \in L$, $x \in X \setminus L$, $y \in \overline{L \cup \{x\}}$, the set $\{v \in \overline{ox} : \overline{vy} \cap L = \emptyset\}$ contains at most one point.

The following proposition is a version of the classical Proclus Parallelity Postulate.

Proposition 13.2 (Proclus Postulate; [1, 3.6.10]). *Let P be a plane in a Proclus liner X and L, L', Λ be three lines in the plane P . If $L \cap L' = \emptyset$ and $|\Lambda \cap L| = 1$, then $|\Lambda \cap L'| = 1$.*

Corollary 13.3 ([1, 3.6.11]). *Let L, Λ be two disjoint lines in a Proclus liner X . If $\|L \cup \Lambda\| \leq 3$, then $|L| = |\Lambda|$.*

14 AFFINE AND PLAYFAIR LINERS

In this section we study the interplay between affine and Playfair liners, and establish that Playfair liners are exactly affine 3-regular 3-long liners.

Theorem 14.1 ([1, 3.7.1]). *For a liner X , the following conditions are equivalent:*

- (1) X is affine and 3-proregular; (2) X is affine and 3-regular;
- (3) for every line $L \subseteq X$ and points $o \in L$, $y \in X \setminus L$, $z \in \overline{L \cup \{y\}} \setminus L$, the set $\{u \in \overline{o y} : \overline{u z} \cap L = \emptyset\}$ is a singleton;
- (4) for every line $L \subseteq X$ and point $p \in X \setminus L$ with $\overline{L \cup \{p\}} \neq L \cup \{p\}$, there exists a unique line Λ such that $p \in \Lambda \subseteq \overline{\{p\} \cup L} \setminus L$.

Theorem 14.2 ([1, 3.7.2]). *For a liner X of cardinality $|X| > 2$, the following conditions are equivalent:*

- (1) X is Playfair; (2) X is affine, 3-regular, and 3-long;
- (3) for every line $L \subseteq X$ and point $x \in X \setminus L$, there exists a unique line Λ such that $x \in \Lambda \subseteq \overline{L \cup \{x\}} \setminus L$.

Corollary 14.3 ([1, 3.7.3]). *Every 3-long affine regular liner X of finite rank r is 2-balanced and has cardinality $|X| = |X|_2^{r-1}$.*

Corollary 14.4 ([1, 3.7.4]). *A liner X with finite lines is Playfair if and only if X is 2-balanced, 3-balanced, and $|X|_3 = |X|_2^2$.*

Proposition 14.5 ([1, 3.7.5]). *A liner X is a Playfair plane if and only if X is 3-long, $\|X\| > 2$, and for every line $L \subseteq X$ and point $x \in X \setminus L$ there exists a unique line L_x in X such that $x \in L_x \subseteq X \setminus L$.*

Theorem 14.6 ([1, 4.1.1]). *Every 4-long affine liner is Playfair and regular.*

The following examples show that Theorem 14.6 does not extend to 3-long affine liners and also to 4-long Proclus liners.

Example 14.7 ([1, 3.7.7+4.5.7]). The ring \mathbb{Z}_{15} endowed with the family of lines $\mathcal{L} = \{x + L : x \in \mathbb{Z}_{15}, L \in \mathcal{B}\}$, where $\mathcal{B} = \{\{0, 1, 4\}, \{0, 6, 8\}, \{0, 5, 10\}\}$ is an example of a 5-parallel non-regular ranked balanced 3-long affine plane.

Question 14.8. *Is every finite balanced affine liner X of rank $\|X\| \geq 4$ regular?*

Example 14.9 ([1, 4.1.8]). For every cardinal κ there exists a non-regular κ -long Proclus liner X of rank $\|X\| = 4$. To construct such a liner, take any $(\kappa + 1)$ -long projective liner Y of rank $\|Y\| = 4$, choose a line $L \subseteq Y$ and a point $x \in L$. Then the subliner $X := Y \setminus (L \setminus \{x\})$ is nonregular, κ -long, Proclus, and has rank $\|X\| = \|Y\| = 4$.

Question 14.10. *Is every 4-long proaffine liner Proclus?*

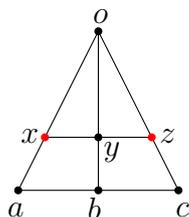
Question 14.11. *Assume that a liner X is κ -parallel and $(3 + \kappa)$ -long for some finite cardinal κ . Is X 3-regular?*

Theorem 14.12 ([1, 4.5.1]). *For a Playfair liner X the following conditions are equivalent:*

- (1) X is regular; (2) X is weakly regular; (3) X is weakly modular;
- (4) X is ranked; (5) X is balanced; (6) X is 4-ranked or 4-balanced;
- (7) X contains no flat $F \subseteq X$ of rank $\|F\| = 4$ and cardinality $|F| = 81$.

By Theorem 14.2, a liner is Playfair if and only if it is affine, 3-regular and 3-long. In fact the first two conditions can be replaced by a single property, called the biaffinity.

Definition 14.13. A liner X is called *biaffine* if for every points $o, a, c \in X$, $b \in \overline{ac}$ and $y \in \overline{ob} \setminus \overline{ac}$, there exist unique points $x \in \overline{oa}$ and $z \in \overline{oc}$ such that $y \in \overline{xz}$ and $\overline{xz} \cap \overline{ac} = \emptyset$.

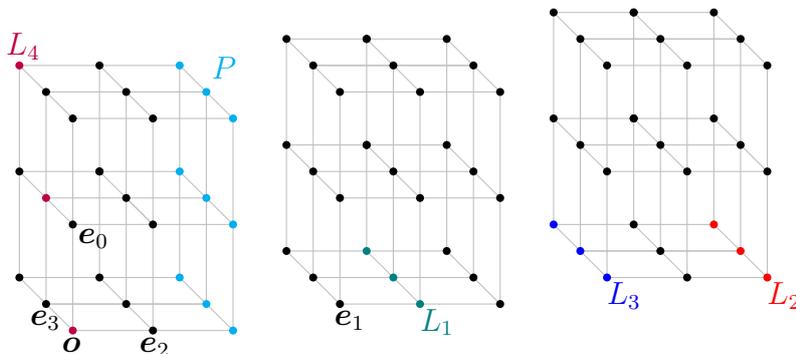


Theorem 14.14 ([1, 4.5.3]). *A liner X is Playfair if and only if X is biaffine and 3-long.*

The following famous example of Marshall Hall [14] shows that Theorem 14.6 cannot be generalized to 3-long affine liners.

Example 14.15 (Hall, 1960; [1, 4.1.11]). There exists a Playfair liner, which is not 4-ranked.

Proof. Let \mathbb{F}_3 be the 3-element field, $X := \mathbb{F}_3^4$ be the vector space over \mathbb{F}_3 with origin o and the standard basis e_0, e_1, e_2, e_3 . Let \mathcal{L} be the family of all 3-elements sets $\{x, y, z\} \subseteq \mathbb{F}_3^4$ such that $x + y + z = (x_1 - y_1)(x_2 y_3 - x_3 y_2)e_0$, where (x_0, x_1, x_2, x_3) and (y_0, y_1, y_2, y_3) are the coordinates of the vectors x, y in the basis e_0, e_1, e_2, e_3 . It can be shown that (X, \mathcal{L}) is a non-regular Playfair liner. This liner contains pairwise disjoint lines $L_1 = e_1 + e_2 + \mathbb{F}_3 e_3$, $L_2 = 2e_1 + 2e_2 + \mathbb{F}_3 e_3$, $L_3 = 2e_1 + \mathbb{F}_3 e_3$ such that the set $\overline{L_1 L_2}$ is a plane, equal to the union of three disjoint lines L_1, L_2 and $L_4 := \mathbb{F}_3(e_0 + e_3)$. On the other hand, the set $\overline{L_1 L_3}$ is the union of the lines L_1, L_2 and the plane $P := \mathbb{F}_3 e_0 + 2e_2 + \mathbb{F}_3 e_3$. Consequently, $\overline{L_1 \cup L_3} = X$ and $\|X\| = 4$. On the other hand, $\{0\} \times \mathbb{F}_3^3$ is a proper 3-dimensional flat in X , witnessing that the liner X is not 4-ranked and hence is not regular. The plane $\overline{L_1 \cup L_2}$ has one-point intersection with the plane $\mathbb{F}_3 e_2 + \mathbb{F}_3 e_4$, which is impossible in regular spaces of rank 4, see Theorem 8.5.



15 HYPERAFFINE, HYPERBOLIC, AND INJECTIVE LINERS

In this section we present some (non-trivial) examples of finite hyperaffine, hyperbolic, and injective liners.

Example 15.1 ([1, 3.8.1]). There exists a hyperaffine non-affine 2-balanced 4-parallel plane with $|X|_2 = 4$ and $|X| = |X|_3 = 25$, see [8, 1.34.1].

Example 15.2 ([1, 3.8.3]). There exists a 4-parallel balanced ranked liner X with $|X|_2 = 4$ and $|X| = |X|_3 = 25$, which is not hyperaffine. It is the group $\mathbb{Z}_5 \times \mathbb{Z}_5$ endowed with the family of lines $\mathcal{L} := \{B + z : B \in \{\{00, 10, 01, 22\}, \{13, 31, 33, 44\}\}, z \in \mathbb{Z}_5 \times \mathbb{Z}_5\}$. The non-hyperaffinity of this liner is witnessed by the points $o := 00$, $x := 41$, $y := 01$, $p := 13$.

Example 15.3 ([1, 3.8.4]). There exists a hyperbolic 8-parallel plane X with $|X|_2 = 7$ and $|X| = |X|_3 = 91$. It is the ring \mathbb{Z}_{91} endowed with the family of lines $\mathcal{L} := \{x + L : L \in \{\{0, 8, 29, 51, 54, 61, 63\}, \{0, 11, 16, 17, 31, 35, 58\}, \{0, 13, 26, 39, 52, 65, 78\}\}, x \in \mathbb{Z}_{91}\}$.

Example 15.4 ([1, 3.8.7]). For every $n \in \{3, 4, 5, 7, 8, 9\}$, there exists an injective 2-balanced $(n^2 - n - 1)$ -parallel plane X with $|X|_2 = n + 1$ and $|X| = |X|_3 = n^3 + 1$.

The injective (and hence hyperbolic) liners appearing in Example 15.4 are classical unitals.

Definition 15.5. A *unital* is a 2-balanced liner X with $|X| = n^3 + 1$ and $|X|_2 = n + 1$ for some finite cardinal $n \geq 2$.

A *classical unital* is the subliner

$$U_q := \{\mathbb{F}_{q^2}(x, y, z) : (x, y, z) \in \mathbb{F}_{q^2}^3 \setminus \{0\}^3 \text{ (} x^{q+1} + y^{q+1} + z^{q+1} = 0)\}$$

of the projective plane $\mathbb{P}\mathbb{F}_{q^2}^3$ over a field \mathbb{F}_{q^2} whose order $|\mathbb{F}_{q^2}| = q^2$ is a square of some number q (which is necessarily a prime power).

Remark 15.6 ([3, 7.42]). Every classical unital U_q is an injective 2-balanced $(q^2 - q - 1)$ -parallel liner with $|U_q|_2 = q + 1$ and $|U_q| = |U_q|_3 = q^3 + 1$. If $q = 2$, then the liner U_q is affine. If $q \geq 3$, then the liner U_q is hyperbolic.

In fact, injective liners are not exotic at all, and as subliners are present in every ω -long plane. We recall that a *plane* is a liner of rank 3.

Theorem 15.7 ([1, 3.8.11]). *Let X be an ω -long plane. For every cardinal $\lambda \in [3, \omega]$, there exists an injective subliner $Y \subseteq X$ such that*

- (1) $|Y| = \|Y\| = \omega$; (2) Y is n -balanced for every positive cardinal n ;
- (3) $|Y|_2 = \lambda$ and $|Y|_n = \omega$ for every cardinal $n \geq 3$.

Question 15.8. *Is every 2-hyperparallel regular liner X hyperaffine? hyperbolic?*

Remark 15.9. Examples 7.4 and 15.2 show that the regularity is essential in Question 15.8.

16 STEINER LINERS

By Theorem 14.6, every 4-long affine liner is regular. If an affine liner X is not 4-long, then all lines in X have the same length $|X|_2 \in \{2, 3\}$. Finite 2-balanced liners X with $|X|_2 = 3$ are well-known in Combinatorics as Steiner Triple Systems, see [8, §II.2]. This motivates the following definition.

Definition 16.1. A liner X is called *Steiner* if every line in X contains exactly three points.

It is easy to see that all Steiner liners are proaffine. Moreover, a Steiner liner is affine if and only if it is hyperaffine if and only if it is injective.

Theorem 15.7 implies the following corollary supplying us with many examples of infinite affine Steiner liners.

Corollary 16.2 ([1, 4.2.4]). *Let $r \in \{3, \omega\}$. Every ω -long plane P contains an infinite affine Steiner subliner X of rank $\|X\| = r$.*

On the other hand, Theorem 10.1 implies the following corollary imposing restrictions on possible cardinalities of finite Steiner liners.

Corollary 16.3 (Kirkman, 1846; [1, 4.2.5]). *Every finite Steiner liner X has cardinality $|X| \in (6\mathbb{N} + 1) \cup (6\mathbb{N} - 3)$.*

Remark 16.4. The necessary condition given by Corollary 16.3 is also sufficient: for every number $n \in (6\mathbb{N} + 1) \cup (6\mathbb{N} - 3)$ there exists a Steiner liner of cardinality n , see [4], [19], [18]. Moreover, if $n \in (6\mathbb{N} + 13) \cup (6\mathbb{N} - 3)$, then there exists an *affine* Steiner liner of cardinality n , see [12].

Theorem 16.5 ([1, 4.2.7]). *Every Steiner 3-regular liner X is p -parallel for some $p \in \{0, 1\}$. Consequently, X is projective or affine.*

Every Steiner liner X carries the commutative binary operation $\circ: X \times X \rightarrow X$ assigning to every pair $(x, z) \in X \times X$ the unique point $y \in X$ such that $\{x, y, z\} = \overline{xz}$. This operation will be called the *midpoint operation* on X . The midpoint operation satisfies the identities

$$x \circ x = x, \quad x \circ y = y \circ x, \quad (x \circ y) \circ y = x,$$

turning X into an involutory idempotent commutative magma.

Now we recall the definitions of some algebraic structures, starting with the most basic notion of a magma. A *magma* is a set X endowed with a binary operation $\cdot: X \times X \rightarrow X$, $\cdot: (x, y) \mapsto xy$. A magma X is

- *commutative* if $xy = yx$ for all $x, y \in X$;
- *associative* if $(xy)z = x(yz)$ for all $x, y, z \in X$;
- *idempotent* if $xx = x$ for all $x \in X$;
- *involutory* if $x(xy) = y$ for all $x, y \in X$;
- *self-distributive* if $x(yz) = (xy)(xz)$ for all $x, y, z \in X$;
- *unital* if there exists an element $e \in X$ such that $xe = x = ex$ for every $x \in X$;
- a *quasigroup* if for every $a, b \in X$ there exist unique $x, y \in X$ such that $ax = b = ya$;
- a *loop* if X is a unital quasigroup;
- a *group* of X is an associative loop;
- *Moufang* if $(xy)(zx) = (x(yz))x = x((yz)x)$ for all $x, y, z \in X$.

A subset S of a magma X is called a *submagma* of the magma X if $\{xy: x, y \in S\} \subseteq S$. A subset S of a loop X is called a *subloop* of X if for every $a, b \in S$, $ab \in S$ and there exist elements $x, y \in S$ such that $ax = b = ya$.

Theorem 16.6 ([1, 4.2.8]). *Any Steiner liner endowed with the midpoint operation is an involutory idempotent commutative magma. Conversely, every involutory idempotent commutative magma (M, \circ) endowed with the family of lines $\mathcal{L} := \{\{x, y, x \circ y\}: x, y \in M \wedge x \neq y\}$ is a Steiner liner. Moreover, a subset $F \subseteq X$ is flat in the liner X if and only if F is a submagma of the magma (X, \circ) .*

A less evident is the relation of Steiner liners to commutative loops. By definition, every loop X contains an element $e \in X$ such that $ex = x = xe$ for every $x \in X$. This element is unique and is called the *identity element* of the loop. A loop X is defined to be of *exponent 3* if $(xx)x = x(xx) = e$ for every element $x \in X$.

For every element e of a Steiner liner X , the binary operation

$$\cdot: X \times X \rightarrow X, \quad \cdot: (x, y) \mapsto xy := e \circ (x \circ y),$$

turns X into a commutative loop of exponent 3 with identity element e . We shall denote this loop by X_e .

Proposition 16.7 ([1, 4.2.10]). *For any set F in a Steiner liner X and any element $e \in F$, the following conditions are equivalent:*

- (1) F is a flat in the liner X ;
- (2) F is a subloop of the loop X_e .

(3) F is a submagma of the magma X_e ; (4) F is a submagma of the magma (X, \circ) .

Theorem 16.8 ([1, 4.2.12]). *For a Steiner liner X the following conditions are equivalent:*

- (1) X is projective; (2) X is 0-parallel; (3) X is 3-balanced with $|X|_3 = 7$;
- (4) for any distinct points $x, y, z \in X$ we have $(x \circ y) \circ (x \circ z) = y \circ z$.

17 HALL LINERS

Definition 17.1. A liner X is called *Hall* if X is 2-balanced and 3-balanced with

$$|X|_3 = (|X|_2)^2 = 9.$$

The liner from Example 14.15 is an example of a Hall liner. Hall liners are well-known in Theory of Combinatorial Designs as *Hall Triple Systems*, see [8, §VI.28]. Corollary 14.4 implies the following characterization of Hall liners.

Theorem 17.2 ([1, 4.4.2]). *A liner X is Hall if and only if X is Steiner and Playfair.*

Theorem 17.3 ([1, 4.4.3]). *For a Steiner liner X , the following conditions are equivalent:*

- (1) X is Hall; (2) the mid-point operation on X is self-distributive;
- (3) for every point $e \in X$, the commutative loop X_e is Moufang.

Theorem 17.4 ([1, 4.4.5]). *For a Hall liner X , the following conditions are equivalent:*

- (1) X is regular; (2) X is weakly regular; (3) X is weakly modular; (4) X is 4-regular;
- (5) X is 4-ranked; (6) X is 4-balanced; (7) X is 4-balanced with $|X|_4 = 27$;
- (8) X contains no flat $F \subseteq X$ of rank $\|F\| = 4$ and cardinality $|F| = 81$;
- (9) for every point $e \in X$, the loop X_e is associative.

A group X is defined to be *nilpotent of class n* for $n \in \mathbb{N}$ if the quotient group X/Z of X by its centre $Z := \{z \in X : \forall x \in X (xz = zx)\}$ is nilpotent of class $n - 1$. The trivial group is nilpotent of class 0. So, non-trivial commutative groups are nilpotent of class 1. A group X is nilpotent of class at most 2 if and only if the quotient group X/Z of X by its centre Z is commutative. It is known ([7], [17]) that every group X of exponent 3 is nilpotent of class at most 3. In particular, for every $m \in \mathbb{N}$, the free Burnside group $B(m, 3) = \langle x_1, \dots, x_m : x^3 = 1 \rangle$ of exponent 3 with m generators is nilpotent of class at most 3. The Burnside group $B(1, 3)$ is nilpotent of class 1 and has cardinality 3, the Burnside group $B(2, 3)$ is nilpotent of class 2 with $|B(2, 3)| = 3^3 = 27$, and for every $m \geq 3$, the Burnside group $B(m, 3)$ is nilpotent of class 3 with $|B(m, 3)| = 3^{C_m^1 + C_m^2 + C_m^3}$.

Example 17.5 ([1, 4.4.7]). Every group X of exponent 3 endowed with the binary operation

$$\circ : X \times X \rightarrow X, \quad \circ : (x, y) \mapsto x \circ y := xy^{-1}x,$$

is a self-distributive idempotent involutory commutative magma. Consequently, X endowed with the family of lines $\mathcal{L} := \{\{x, y, xy^{-1}x\} : x, y \in X \wedge x \neq y\}$ is a Hall liner. This Hall liner is regular if and only if the group X is nilpotent of class at most 2.

Remark 17.6. By a deep result of Bruck and Slaby [5, Theorem 10.1], every finitely generated commutative Moufang loop X of exponent 3 is centrally nilpotent, which implies that $|X| = 3^n$ for some $n \in \mathbb{N}$. This algebraic result implies that every Hall liner X of finite rank has cardinality $|X| = 3^n$ for some $n \in \mathbb{N}$. This fact was also proved by Hall [13] by group-theoretic methods, without involving the machinery of Moufang loops. Theorem 17.4 implies that all non-regular Hall liners of rank 4 are isomorphic and have cardinality 81. The

following table taken from [8, §VI.28.10] shows the number \bar{h} of non-isomorphic non-regular Halls liners of cardinality 3^n .

n	1	2	3	4	5	6	7	8	9
\bar{h}	0	0	0	1	1	3	12	≥ 45	?

The liner in Example 14.15 is the unique non-regular Hall liner of cardinality 81.

18 SUBPARALLELITY

Definition 18.1. Given two flats A and B in a liner, we write $A \parallel B$ and say that A is *subparallel to B* if $\forall a \in A$ ($A \subseteq \overline{\{a\} \cup B}$).

Remark 18.2. For two intersecting flats A, B in a liner, we have $A \parallel B$ if and only if $A \subseteq B$.

It can be shown that a flat A in a liner X is subparallel to a flat $B \subseteq X$ if and only if for every $a, b \in A$ the flat \overline{ab} is subparallel to B .

Theorem 18.3 ([1, 6.1.4]). *Let X be a κ -ranked liner for some cardinal κ . For two flats $A, B \subseteq X$ with $\|B\| < \kappa$, the following conditions are equivalent:*

- (1) $A \parallel B$; (2) either $A \subseteq B$ or $A \subseteq \overline{\{a\} \cup B} \setminus B$ for some $a \in A$;
- (3) either $A \subseteq B$ or $A \cap B = \emptyset$ and $\|A\|_B = 1$.

If $\|B\| < \omega$, then the conditions (1)–(3) are equivalent to the condition

- (4) either $A \subseteq B$ or $A \cap B = \emptyset$ and $\|A \cup B\| = \|B\| + 1$.

Corollary 18.4 ([1, 6.1.5]). *Let κ be a finite cardinal and A, B are two flats in a κ -ranked liner X . If $\|A\| = \|B\| < \kappa$, then $A \parallel B \Leftrightarrow B \parallel A$.*

Corollary 18.5 ([1, 6.1.7]). *Let A, B be two flats in a $\|B\|$ -ranked liner X such that $A \parallel B$ and $\|A\| = \|B\| < \omega$. If $A \cap B \neq \emptyset$, then $A = B$.*

Proposition 18.6 ([1, 6.1.8]). *Let κ be a cardinal and A, B be two flats in a κ -ranked liner X . If $\|B\| < \kappa$, $A \parallel B$, and $B \neq \emptyset$, then $\|A\| \leq \|B\|$.*

Remark 18.7. The non-ranked liner in Example 3.5 contains two lines L, Λ such that $L \parallel \Lambda$ but not $\Lambda \parallel L$.

19 PARALLELITY

Definition 19.1. Given two flats A and B in a liner X , we write $A \parallel B$ and say that A and B are *parallel* if A is subparallel to B and B is subparallel to A . Therefore,

$$A \parallel B \Leftrightarrow (A \parallel B \wedge B \parallel A).$$

For two flats A, B , the negation of $A \parallel B$ is denoted by $A \not\parallel B$.

Theorem 18.3 implies the following characterization.

Theorem 19.2 ([1, 6.2.2]). *Let κ be a cardinal, X be a κ -ranked liner, and A, B be two nonempty flats in X . If $\max\{\|A\|, \|B\|\} < \kappa$, then the following conditions are equivalent:*

- (1) $A \parallel B$;
- (2) either $A = B$ or $A \subseteq \overline{\{a\} \cup B} \setminus B$ and $B \subseteq \overline{\{b\} \cup A} \setminus A$ for some points $a \in A, b \in B$;
- (3) either $A = B$ or else $A \cap B = \emptyset$ and $\|A\|_B = 1 = \|B\|_A$.

If $\min\{\|A\|, \|B\|\} < \omega$, then the conditions (1)–(3) are equivalent to the condition

- (4) either $A = B$ or else $A \cap B = \emptyset$ and $1 + \|A\| = \|A \cup B\| = 1 + \|B\|$.

Remark 19.3. Any two singletons in a liner are parallel.

Proposition 18.6 implies

Corollary 19.4 ([1, 6.2.4]). *Let κ be a cardinal, X be a κ -ranked liner and A, B be two flats in X . If $0 < \min\{\|A\|, \|B\|\} < \kappa$ and $A \parallel B$, then $\|A\| = \|B\|$.*

Proposition 19.5 ([1, 6.2.5]). *Two intersecting flats in a liner are parallel if and only if they are equal.*

Theorem 19.2 implies the following characterization of the parallelity of lines in 3-ranked liner, which often is taken as the definition of the parallelity of lines.

Corollary 19.6 ([1, 6.2.6]). *Two lines L, Λ in a 3-ranked liner are parallel if and only if $\|L \cup \Lambda\| \leq 3$ and either $L = \Lambda$ or $L \cap \Lambda = \emptyset$.*

Corollary 19.7 ([1, 6.2.7]). *Let A, B, C be three lines in a Proclus liner X . If $A \parallel B$, $B \parallel C$ and $\|A \cup C\| \leq 3$, then $A \parallel C$.*

20 PARALLELITY IN (WEAKLY) MODULAR LINERS

Proposition 20.1 ([1, 6.3.1]). *For two flats A, B in a modular liner, the following conditions are equivalent:*

- (1) $A \parallel B$; (2) $A \subseteq B$ or $\|A\| = 1$. (3) $A \subseteq B$ or $\|A\| \leq 1$.

Corollary 20.2 ([1, 6.3.2]). *For two flats A, B in a modular liner, the following conditions are equivalent:*

- (1) $A \parallel B$; (2) $A = B$ or $\max\{\|A\|, \|B\|\} = 1$; (3) $A = B$ or $\max\{\|A\|, \|B\|\} \leq 1$.

Proposition 20.3 ([1, 6.4.1]). *Let A, B, C be flats in a weakly modular liner. If $A \parallel B$ and $B \cap C \neq \emptyset$, then $A \cap C \parallel B \cap C$.*

Corollary 20.4 ([1, 6.4.2]). *Let A, B, C be flats in a weakly modular liner. If $A \parallel B$ and $A \cap C \neq \emptyset \neq B \cap C$, then $A \cap C \parallel B \cap C$.*

Proposition 20.5 ([1, 6.4.3]). *For two disjoint flats A, B in a weakly modular liner, the following conditions are equivalent:*

- (1) $A \parallel B$; (2) for every $b \in B$, the flats A and $B \cap \overline{\{b\} \cup A}$ are parallel;
- (3) there exists a flat $C \subseteq B$ such that $A \parallel C$; (4) $\forall a, x \in A \forall b \in B \exists y \in B (\overline{ax} \parallel \overline{by})$.

21 PARALLELITY IN PROAFFINE REGULAR LINERS

Theorem 21.1 ([1, 6.5.1]). *For a regular liner X , the following conditions are equivalent:*

- (1) the liner X is proaffine;
- (2) for every flat $A \subseteq X$ and points $a \in A$, $b \in X \setminus A$, $p \in \overline{A \cup \{b\}} \setminus A$, there exists a point $u \in \overline{ab}$ such that $\forall v \in \overline{ab} \setminus \{u\} (\overline{vp} \cap A \neq \emptyset)$;
- (3) for every flat $A \subseteq X$ and point $b \in X \setminus A$ there exists at most one flat $B \subseteq X$ such that $b \in B$ and $B \parallel A$.

In proaffine regular liners, the parallelity is an equivalence relation (which is a generalization of Proposition I.30 on parallel lines in Euclid's "Elements" [11]).

Theorem 21.2 ([1, 6.5.4+6.5.6]). *For any flats A, B, C in a proaffine regular liner X we have the implications $(A \parallel B \wedge B \parallel C) \Rightarrow (A \parallel C)$ and $(A \parallel B \wedge B \parallel C) \Rightarrow (A \parallel C)$.*

Example 21.3. The Hall liner in Example 14.15 contains lines L_1, L_2, L_3 such that $L_1 \parallel L_2 \parallel L_3$ but $L_1 \not\parallel L_3$.

The following theorem generalizes the Proclus Parallelity Postulate 13.2.

Theorem 21.4 ([1, 6.5.8]). *Let A, B be parallel flats in a proaffine regular liner X and L be a line in the flat $\overline{A \cup B}$. If $|L \cap A| = 1$, then $|L \cap B| = 1$.*

Theorem 21.5 ([1, 6.5.9]). *A nonempty flat A in a proaffine regular liner X is subparallel to a flat B in X if and only if there exists a point $a \in A$ and a set $\Lambda \subseteq A$ such that $A = \overline{\Lambda}$ and $\forall x \in \Lambda (\overline{ax} \parallel B)$.*

22 PARALLELITY IN AFFINE AND PLAYFAIR LINERS

Theorem 22.1 ([1, 6.6.1]). *For a regular liner X , the following conditions are equivalent:*

- (1) *the liner X is affine;*
- (2) *for every flat $A \subseteq X$ and points $a \in A, b \in X \setminus A, p \in \overline{A \cup \{b\}} \setminus A$, there exists a unique point $u \in \overline{ab}$ such that $\overline{up} \cap A = \emptyset$;*
- (3) *for every flat $A \subseteq X$ and points $a \in A, b \in X \setminus A$ with $\overline{ab} \neq \{a, b\}$, there exists a unique flat $B \subseteq X$ such that $b \in B$ and $B \parallel A$.*

Theorem 22.2 ([1, 6.6.2]). *For every line L in a Playfair liner X and every point $x \in X \setminus L$, there exists a unique line Λ in X such that $x \in \Lambda$ and $\Lambda \parallel L$.*

For regular Playfair liners, we have a higher dimensional counterpart of Theorem 22.2, which follows from Theorem 22.1.

Corollary 22.3 ([1, 6.6.3]). *For any flat A in a regular Playfair liner X and every point $x \in X$, there exists a unique flat $B \subseteq X$ such that $x \in B$ and $B \parallel A$.*

Let us recall that a liner X is *Bolyai* if for every plane $P \subseteq X$, line $L \subseteq P$, and point $x \in P \setminus L$, there exists a line Λ in X such that $x \in \Lambda \subseteq P \setminus L$.

Theorem 22.4 ([1, 6.6.4]). *For every liner X the following conditions are equivalent:*

- (1) *X is Playfair and regular;*
- (2) *X is Bolyai, 3-ranked, and for every lines $A, B, C \subseteq X$, if $A \parallel B$ and $B \parallel C$, then $A \parallel C$.*

Proposition 22.5 ([1, 6.6.5]). *For any parallel lines A, B, C in a Playfair liner X , the flat $\overline{A \cup B \cup C}$ is a regular subliner of X .*

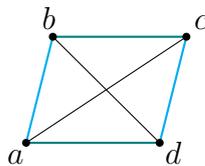
23 TRIANGLES AND PARALLELOGRAMS IN LINERS

A triple of points $abc \in X^3$ in a liner X is called a *triangle* in X if $\|\{a, b, c\}\| = 3$. The points a, b, c are called the *vertices* of the triangle abc , and the lines $\overline{ab}, \overline{bc}, \overline{ca}$ are called the *sides* of the triangle abc .

A quadruple of points $abcd \in X^4$ in a liner X is called a *parallelogram* in X if

$$\overline{ab} \parallel \overline{cd} \neq \overline{ab} \quad \text{and} \quad \overline{bc} \parallel \overline{ad} \neq \overline{bc}.$$

The points a, b, c, d are called the *vertices*, the lines $\overline{ab}, \overline{bc}, \overline{cd}, \overline{da}$ are called the *sides*, and the lines $\overline{ac}, \overline{bd}$ are called the *diagonals* of the parallelogram $abcd$.



Corollary 19.6 implies the following characterization of parallelograms in 3-ranked liners.

Proposition 23.1 ([1, 6.7.4]). *A quadruple of points $abcd$ in a 3-ranked liner X is a parallelogram if and only if $\overline{a} \overline{b} \cap \overline{c} \overline{d} = \emptyset = \overline{b} \overline{c} \cap \overline{a} \overline{d}$ and $\|\{a, b, c, d\}\| \leq 3$.*

Proposition 23.2 ([1, 6.7.7]). *If a liner X is κ -parallel for some finite nonzero cardinal κ , then for every triangle abc there exists a point $d \in X$ such that $abcd$ is a parallelogram.*

Theorem 23.3 ([1, 6.7.8]). *A liner X is Playfair if and only if X is affine, 3-ranked, and for any triangle abc in X there exists a unique point $d \in X$ such that $abcd$ is a parallelogram.*

24 BOOLEAN PARALLELOGRAMS AND BOOLEAN LINERS

A parallelogram $abcd$ in a liner X is called *Boolean* if its diagonals $\overline{a} \overline{c}$ and $\overline{b} \overline{d}$ are parallel.

For 3-ranked Steiner affine liners, Proposition 23.2 and Theorem 23.3 are completed by the following proposition.

Proposition 24.1 ([1, 6.8.2]). *For any triangle abc in a 3-ranked Steiner affine liner X , there exists a unique point $d \in X$ such that $abcd$ is a non-Boolean parallelogram.*

Corollary 24.2 ([1, 6.8.3]). *For every triangle abc in a 3-long 3-ranked affine liner X , there exists a point $d \in X$ such that $abcd$ is a parallelogram.*

Corollary 24.3 ([1, 6.8.4]). *A Steiner liner is Playfair if and only if it is affine, 3-ranked, and contains no Boolean parallelograms.*

Let us recall that a liner is *line-finite* if all its lines are finite sets. Observe that a 2-balanced liner X is line-finite if and only if the cardinal $|X|_2$ is finite.

Proposition 24.4 ([1, 6.8.5]). *If a line-finite 2-balanced 3-balanced 3-ranked liner X has $|X|_3 \geq |X|_2^2$ (and $|X|_3 > 3 \cdot |X|_2^2 - 9 \cdot |X|_2 + 9$), then for every triangle abc there exists a point $d \in X$ such that $abcd$ is a (Boolean) parallelogram.*

Corollary 24.5 ([1, 6.8.6]). *For every triangle abc in a Steiner ranked plane X of cardinality $|X| > 9$, there exists a point $d \in X$ such that $abcd$ is a Boolean parallelogram.*

Definition 24.6. A liner X is defined to be *Boolean* if for any quadruple $abcd \in X^4$, $\overline{a} \overline{b} \cap \overline{c} \overline{d} = \emptyset = \overline{b} \overline{c} \cap \overline{a} \overline{d}$ implies $\overline{a} \overline{c} \cap \overline{b} \overline{d} = \emptyset$.

Theorem 24.7 ([1, 6.8.8]). *A 3-ranked liner X is Boolean if and only if every parallelogram in X is Boolean.*

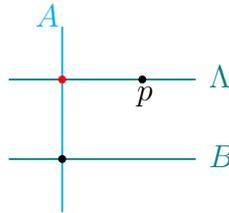
Example 24.8. Let V be a vector space over a field F . The space V endowed with its canonical line structure is a Boolean liner if and only if the field F has characteristic two.

25 HYPER-BOLYAI LINERS

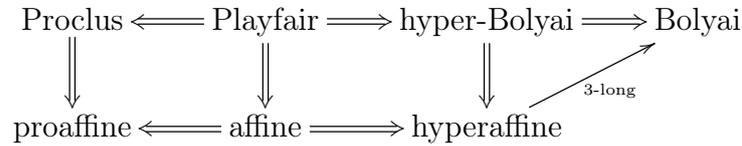
Motivated by the problem of extending triangles to parallelograms, we introduce and study in this section the class of hyper-Bolyai liners, intermediate between the Playfair and Bolyai liners.

Let us recall that a liner X is *Bolyai* if for any plane $P \subseteq X$, line $L \subseteq P$ and point $p \in P \setminus L$ there exists a line Λ such that $p \in \Lambda \subseteq P \setminus L$.

Definition 25.1. A liner X is defined to be *hyper-Bolyai* if for any plane $P \subseteq X$, concurrent lines $A, B \subseteq P$, and point $p \in P \setminus B$, there exists a line $\Lambda \subseteq X$ such that $p \in \Lambda$, $\Lambda \parallel B$, and $|\Lambda \cap A| = 1$.



Hyper-Bolyai liners relate to some other types of liners as follows.



Non-trivial implications in this diagram are established in Theorems 13.1, 14.2, and the following proposition.

- Proposition 25.2** ([1, 6.9.2]). (1) *Every hyper-Bolyai liner is Bolyai.*
 (2) *Every hyper-Bolyai liner is hyperaffine.* (3) *Every hyperaffine 3-long is Bolyai.*
 (4) *A liner is Playfair if and only if it is Proclus and hyper-Bolyai.*

Remark 25.3. By Example 15.2, there exists a 4-long Bolyai plane, which is not hyperaffine. By Example 15.1, there exists a non-Playfair hyperaffine 4-long ranked plane, which is hyper-Bolyai.

The following theorem characterizes hyper-Bolyai liners, in the spirit of the characterizations of Proclus and Playfair liners given in Theorems 13.1 and 14.2.

Theorem 25.4 ([1, 6.6.5]). *A liner X of cardinality $|X| > 2$ is hyper-Bolyai if and only if X is 3-long, hyperaffine, and 3-ranked.*

The following proposition (which motivated our study of hyper-Bolyai liners) show that triangles in hyper-Bolyai liners can be extended to parallelograms.

Proposition 25.5 ([1, 6.9.6]). *For every triangle abc in a hyper-Bolyai liner X , there exists a point $d \in X$ such that $abcd$ is a parallelogram.*

Remark 25.6. By Example 15.2, there exists a 4-parallel plane X with $|X|_2 = 4$ and $|X| = |X|_3 = 25$, which is not hyperaffine and hence not hyper-Bolyai. By Proposition 23.2, every triangle in X can be completed to a parallelogram. This example shows that Proposition 25.5 cannot be turned into a characterization of hyper-Bolyai liners.

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